Math 361, Fall 2015 **Jerry L. Kazdan**

Problem Set 9

Due: Thurs. Nov. 19 in class. [Late papers will be accepted (without penalty) until 1:00 PM Friday.]

Please carefully read Chapter 9 in the Marsden-Hoffman text.
The following short answer problems from Marsden-Hoffman are not assigned, but you should know how to do them. Some short answer problems will be on our exams.

p. 466 #3, p. 469 #2, p, 490 #5, p. 505 #4, p. 508 #3, p. 508 #5, p. 512 #5

1. In the plane $\mathbb{R}^2$, prove that the points on the graph of the curve $y = x^2$ have measure zero.

2. Determine if the following improper integrals converge or converge absolutely. Justify your assertions.

   a) $\displaystyle\int_1^\infty \frac{\sin x}{1+x}\,dx$ b) $\displaystyle\int_1^\infty \frac{\sin x}{1+x^2}\,dx$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(x) \ge 0$. If the improper integral $\int_0^\infty f(x)\,dx$ exists, then $\lim_{x\to\infty} f(x) = 0$. Proof or counterexample.

4. [Marsden-Hoffman, p. 493 #35] Let $A_n = [(n+1) + (n+2) + \cdots + (n+n)]/n$. Prove that $\lim_{n\to\infty} A_n/n = 3/2$. [Suggestion: Use the Riemann integral.]

5. [Marsden-Hoffman, p. 494 #41] Let $\mathcal{R} = \{f : [0,1] \to \mathbb{R} \mid f \text{ is Riemann Integrable }\}$ and define
$$d(f,\,g) = \int_0^1 |f(x) - g(x)|\,dx.$$
Is d a metric on the space $\mathcal{R}$? Why?

6. [Marsden-Hoffman p. 495 #47] For every $\alpha > 0$, compare $\int_0^N x^\alpha\,dx$ with $\sum_{n=1}^N n^\alpha$ and $\sum_{n=0}^{N-1} n^\alpha$ and hence determine
$$\lim_{N\to\infty} \frac{1}{N^{1+\alpha}} \sum_{n=1}^N n^\alpha.$$

7. Let $f : [a,\,b] \to \mathbb{R}]$ be an invertible continuous function, $y = f(x)$ and let $\int_a^b f(x)\,dx = F(b) - F(a)$, so $F' = f$. Say $x = \varphi(y)$ is the inverse function of f. Let $\alpha = F(a)$ and $\beta = F(b)$.

a) Show that

$$\int_\alpha^\beta \varphi(y)\, dy + \int_a^b f(x)\, dx = bf(b) - af(a).$$

Equivalently,

$$\int_\alpha^\beta \varphi(y)\, dy = bf(b) - af(a) - [F(b) = F(a)].$$

Thus, if you can explicitly integrate a function, you can also integrate the inverse function.

b) Apply this to the examples $y = x^2$ and $y = e^x$.

8. Let $f(x) \in C([a, b])$. Show that

$$\exp\left[\frac{1}{b-a}\int_a^b f(x)\, dx\right] \le \frac{1}{b-a}\int_a^b \exp[f(x)]\, dx$$

Similarly, let $\mathcal{D} \in \mathbb{R}^2$ be a bounded region whose boundary is smooth enough that the integrals below all exist. Define the *average*, $\bar{h}$, of a continuous function h by $\bar{h} = \frac{1}{\text{Area}(\mathcal{D})}\iint_{\mathcal{D}} h\, dA$, where dA is the usual element of area. Show that

$$\frac{1}{\text{Area}(\mathcal{D})}\iint_{\mathcal{D}} e^h\, dA \ge e^{\bar{h}}$$

9. If A is a positive definite 2×2 matrix and $b \in \mathbb{R}^2$ is any vector, show that the improper integral

$$\iint_{R^2} e^{-[\langle x,\, Ax\rangle + \langle b,\, x\rangle]}\, dx\, dy$$

exists. [Challenge: find an explicit formula for this in terma of A and b.]

10. a) Compute

$$\iint_{\mathbb{R}^2} \frac{dx\, dy}{(1 + 4x^2 + 9y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx\, dy}{(1 + x^2 + 2xy + 5y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx\, dy}{(1 + 5x^2 - 4xy + 5y^2)^2}.$$

b) Compute $\displaystyle\iint_{\mathbb{R}^2} \frac{dx_1\, dx_2}{[1 + \langle x,\, Cx\rangle]^2}$, where C is a positive definite (symmetric) 2×2 matrix, and $x = (x_1, x_2) \in \mathbb{R}^2$.

11. [Generalization of the previous problem] Let $h(t)$ be a given function and say you know that $\int_0^\infty h(t)\, dt = \alpha$.

a) If C be a positive definite 2×2 matrix. Show that

$$\iint_{\mathbb{R}^2} h(\,\langle x,\, Cx\rangle\,)\; dA = \frac{\pi\alpha}{\sqrt{\det C}}.$$

b) Generalize this to obtain a formula for

$$\iint_{\mathbb{R}^n} h(\,\langle x,\, Cx\rangle\,)\; dV,$$

where now C be a positive definite $n \times n$ matrix. The answer will involve some integral involving h and also the "area" of the unit sphere S^{n-1} in $\mathbb{R}^n$.

12. a) Compute $\displaystyle\iint_{\mathbb{R}^2} e^{-(5x^2-4xy+5y^2)}\, dx\, dy.$ b) Compute $\displaystyle\iint_{\mathbb{R}^2} e^{-(5x^2-4xy+5y^2-2x+3)}\, dx\, dy.$

[Last revised: November 17, 2015]