Math 361, Fall 2015 **Jerry L. Kazdan**

Problem Set 8

DUE: Thurs. Nov. 5 in class. [Late papers will be accepted (without penalty) until 1:00 PM Friday.]

Note: Exam 2 is on Tuesday, Nov. 10.

Please carefully read Sections 8.1–8.5 in the Marsden-Hoffman text.

1. Compute $\int_0^c x^3\,dx$ with your bare hands by using a Riemann sum, partitioning the interval $0 \le x \le c$ into sub-intervals of equal length. You may use without proof that
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

2. Compute $\int_0^b \sin x\,dx$ by using a Riemann sum, partitioning the interval $0 \le x \le b$ into sub-intervals of equal length. You may use without proof that
$$\sin x + \sin 2x + \cdots + \sin nx = \frac{\cos\frac{x}{2} - \cos(n+\frac{1}{2})x}{2\sin\frac{x}{2}}.$$
[One way to get this is by summing the geometric series $e^{ix} + e^{i2x} + \cdots + e^{inx}$ and taking the imaginary part].

3. [Marsden-Hoffman, p. 456 #2–3] On the set $A = [0,1]\times[0,1] \subset \mathbb{R}^2$, let $f(x,y) = 1$ for all $(x,y) \in A$, $x \neq 0$ and $f(0,y) = 0$. Prove that f is Riemann integrable on this set and compute its value.

4. [Marsden-Hoffman, p. 454 #1]. Show that the "volume" of the set $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is zero.

5. [Marsden-Hoffman, p. 454 #2]. Show that the three dimensional measure of the xy plane in $\mathbb{R}^3$ is zero.

6. [Marsden-Hoffman, p. 454 #5]. Must the boundary of a set of measure zero have measure zero? Proof or counterexample.

7. Let f be continuous in the square $Q := \{[0, 1]\times[0, 1] \subset \mathbb{R}^2\}$ and assume that $f(x,y) \geq 0$ for all points in Q. Use the definition of the integral as a Riemann sum to show that if $\int_Q f(x,y)\,dx\,dy = 0$, then $f(x,y)) = 0$ everywhere. [You will need to use that since f is continuous, if it is positive at some point, then it is positive in some rectangle containing the point.]

   This may be false if f is discontinuous somewhere. Example?

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map $T : (u,v) \mapsto (x(u,v),\, y(u,v))$. Assume that it is an invertible map from the set A to the set B, and that $DT(u,v)$ is invertible at every point of A. Let $f(x,y)$ be a smooth function for all points $(x,y) \in B$ and let

$$g(u,v) = f(x(u,v),\, y(u,v))$$

   (perhaps think of T as a smooth change of coordinates).

   a) Show that the point $x = p$, $y = q$ in B is a critical point of a (smooth) function $f(x,y)$ if and only if (p,q) is the image of critical point of $(a,\, b)$ of $g(u,v)$.

   b) Say $(a,\, b)$ is a *non degenerate* critical point of $g(u,v)$ so $p = x(a,b)$, $q == y(a,b)$ is a critical point of f. Show that $(p,\, q)$ is a non-degenerate critical point of f and that it is a local minimum if and only if (a,b) is a local minimum of g. [If it helps, you may assume that the Hessian $g''(a,\, b)$ is a diagonal matrix.]

      [MORAL: if you think of the map T as a change of coordinates, this says that T preserves both the critical points of g and if they are local maxima, minima, or saddles.]

[Last revised: November 1, 2015]