Math 361, Fall 2015 **Jerry L. Kazdan**

Problem Set 7

DUE: Thurs. Oct. 29 in class. [Late papers will be accepted (without penalty) until 1:00 PM Friday.]

Note: The date of Exam 2 has been changed to Tuesday, Nov. 10.

Please carefully read Sections 8.1–8.4 in the Marsden-Hoffman text.

The following short answer problems from Marsden-Hoffman are not assigned, but you should know how to do them. Some short answer problems will be on our exams.

p. 413 #1, 2, 3,   p. 444 #35

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth map.
   a) If $\|\nabla f(x)\| \le M$ everywhere, show that $\|f(x) - f(y)\| \le M\|x - y\|$.
   b) Let A be the annular region $A := \{x \in \mathbb{R}^2 : 1 < \|x\| < 2\}$ and $f : A \to \mathbb{R}$ a smooth map. If $\|\nabla f(x)\| \le M$ for all points in A, estimate $\|f(x) - f(y)\|$ for x and y in A.

2. PROOF OR COUNTEREXAMPLE? There is *no* smooth function defined on $\mathbb{R}^2$ with exactly two critical points, both non-degenerate local minima.

3. If $h(x, y) = x^2 - 2xy + 5y^2$, since then $h(x, y) = (x - y)^2 + 4y^2$, it is clear that under the change of coordinates $u = x - y$, $v = 2y$ we can write $h = u^2 + v^2$ as a sum of squares.

   Prove (with your bare hands, *not* using the Morse Lemma) that one can do this near the origin for any smooth function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ with the properties that

   $$f(0, 0) = 0, \qquad f'(0, 0) = 0, \qquad f''(0, 0) \text{ is positive definite.}$$

   [Here f' is the gradient and f'' the second derivative matrix.]

4. a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function with the properties that $f''(x) \ge 0$ and $f(x) \le C$ for all $x \in \mathbb{R}$. Show that $f(x) =$ constant.
   b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with the properties that the hessian matrix $f''(x)$ is positive semi-definite and that $f(x) \le C$ for all $x \in \mathbb{R}^2$. Does this imply that $f(x) =$ constant? Proof or counterexample.

5. [Marsden-Hoffman, p. 439 #6] Determine whether the "curve" described by the equation $x^2 + y + \sin(xy) = 0$ can be written in the form $y = f(x)$ in a neighborhood of $(0, 0)$.

   Does the implicit function theorem allow you to say weather the equation can be written in the form $x = h(y)$ in some neighborhood of $(0, 0)$?

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x,y) = (u(x,y),\, v(x,y))$ and assume that u and v satisfy the *Cauchy-Riemann equations*

$$u_x(x,y) = v_y(x,y) \qquad \text{and} \qquad u_y(x,y) = -v_x(x,y).$$

   a) Show that this map is invertible near a point (x,y) if and only if $Df(x,y) \neq 0$.
   b) Show that the inverse map also satisfies the Cauchy-Riemann equations.

7. [Marsden-Hoffman p. 420 #4] Find the extrema of $f(x,y,z) = x + y + z$ subject to the constraints: $x^2 + y^2 = 1$, $2x + z = 1$.

8. [Marsden-Hoffman p. 444 #35] Find the relative extrema of $f(x,y) = x^2 + y^2$ subject to the constraint $x^2 - y^2 = 1$.

9. [Marsden-Hoffman p. 444 #38] A rectangular box with no top is to have a surface area of 16 square meters. Find the dimensions that maximize the volume.

10. Let A be a real square symmetric matrix and let $v \in \mathbb{R}^n$ be a point on the unit sphere, $\|x\| = 1$, where $f(x) = \langle x,\, Ax\rangle$ has its maximum. Show that v is an eigenvector of A. What is the corresponding eigenvalue?

11. [CONTINUATION OF THE PREVIOUS PROBLEM] Say $w \in \mathbb{R}^n$ is a point maximizing $f(x)$ in the set $\|x\| = 1$ with w also perpendicular to v, so $\langle w,\, v\rangle = 0$. Show that w is also an eigenvector of A.

Bonus Problem

[Please give these directly to Professor Kazdan]

B-1 a) If $u(x_1, x_2, \ldots, x_n)$ is a given smooth function, let $u'' := \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ be its second derivative (Hessian) matrix. Find all solutions of $\det(u'') = 1$ in the special case where $u = u(r)$ depends only on $r = \sqrt{x_1^2 + \cdots x_n^2}$, the distance to the origin.

   b) Let $x = (x_1, x_2, \ldots, x_n)$ and A be a square matrix with $\det A = 1$. If $u(x)$ satisfies $\det(u'') = 1$ (see above), and $v(x) := u(Ax)$, show that $\det(v'') = 1$ also. [Remark: the differential operator $\det(u'')$ is interesting because its symmetry group is so large.]

[Last revised: October 29, 2015]