Math 361, Fall 2015 **Jerry L. Kazdan**

Problem Set 6

DUE: Thurs. Oct. 22 in class. [Late papers will be accepted (without penalty) until 1:00 PM Friday.]

Note: The date of Exam 2 has been changed to Tuesday, Nov. 10.

Please carefully read Chapter 7.1-7.2, 7.5-7.7 in the Marsden-Hoffman text.

The following short answer problems from Marsden-Hoffman are not assigned, but you should know how to do them. Some short answer problems will be on our exams.

p. 396,#1, 2, 3, 5, p. 400 #1, 2, 5, p. 411 #1, p. 413 #1, 2, 3, p. 444 #35

1. a) Find a function $u(y, s)$ for $(y, s) \in \mathbb{R}^2$ such that $u_s = 0$ and $u(y, 0) = \sin 2y$.

   b) Say we want to solve the partial differential equation $u_x(x, t) - 2u_t(x, t) = 0$. One approach is to seek some change of variable $y = ax + bt$, $s = cx + dt$ (where a, b, c, and d are constants) so that in the y, s variables this equation has the simpler form $u_s = 0$ and then use the previous part. Use this approach to solve $u_x - 2u_t = 0$ with $u(x, 0) = x^2$.

2. Let $w(x)$ and $u(x, y)$ be given smooth functions of the real variables x and y.

   a) If $w(x)$ satisfies $w'' - c(x)w = 0$, where $c(x) > 0$ is a given function, show that w cannot have a local positive maximum (that is, a local maximum where $w > 0$). Also show that w cannot have a local negative minimum.

   b) If $u(x, y)$ satisfies $4u_{xx} + 3u_{yy} - 5u = 0$, show that it cannot have a local positive maximum. Also show that u cannot have a local negative minimum.

   c) If a function $u(x, y)$ satisfies $4u_{xx} + 3u_{yy} - 5u = 0$ in a bounded region $\mathcal{D} \in \mathbb{R}^2$ and is zero on the boundary of the region, show that $u(x, y)$ is zero throughout the region.

3. The following equations define a map $F : (x, y, z) \mapsto (u, v, w)$:

$$u(x, y, z) = x + xyz^2$$
$$v(x, y, z) = y + xy$$
$$w(x, y, z) = z + cx + 3z^2$$

   Clearly $F : (1, 1, 0) \mapsto (1, 2, c)$. Write $p = (1, 1, 0)$ and $q = (1, 2, c)$.

   a) Compute the derivative $F'(p)$.

   b) For which value(s) of the constant c can the system of equations: can be solved for x, y, z as smooth functions of u, v, w near $(1, 1, 0)$? Justify your assertion(s).

c) If c is one of these "good" values, let $G : (u, v, w) \mapsto (x, y, z)$ be the map inverse to F. Compute the derivative $G'(q)$ and use it to compute $\partial y(u, v, w)/\partial v$ at q.

4. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 =& 0 \\ x - y + 2z + u =& 0 \\ 2x + 2y - 3z + 2u =& 0 \end{aligned}$$

can be solved for x, y, u in terms of z, for x, z, u in terms of y for y, z, u in terms of x, but *not* for x, y, z in terms of u.

5. Let $p(x) := (x-1)(x-2)\cdots(x-6) = x^6 - 21x^5 + \cdots$ and let $p(x,t)$ be the perturbed polynomial obtained by replacing $-21x^5$ by $-(21+t)x^5$ (think of t as being small). Let $x(t)$ denote the perturbed value of the root $x = 4$, so $x(0) = 4$.

   a) Show that $x(t)$ is a smooth function of t for all $|t|$ sufficiently small.

   b) Compute the sensitivity of this root as one changes t, that is, compute $dx(t)/dt\big|_{t=0}$.

6. [Marsden-Hoffman p. 413 #4] Let $f(x, y) = x^2 + y^2 + 3y^3 + 8x^4 + x^2 e^x \sin x + 6$. Show there are new coordinated u,v, where $u = u(x, y)$, $v = v(x, y)$ for which

$$f(x, y) = u^2 + v^2 + 6.$$

in a neighborhood of the origin.

Problems 7, 8, and 9 below are deferred until Homework Set 7

7. [Marsden-Hoffman p. 420 #4] Find the extrema of $f(x, y, z) = x + y + z$ subject to the constraints: $x^2 + y^2 = 1$, $2x + z = 1$.

8.]Marsden-Hoffman p. 444 #35] Find the relative extrema of $f(x, y) = x^2 + y^2$ subject to the constraint $x^2 - y^2 = 1$.

9. Let A be a real square symmetric matrix and let $v \in \mathbb{R}^n$ be a point on the unit sphere, $\|x\| = 1$, where $f(x) = \langle x, Ax \rangle$ has its maximum. Show that v is an eigenvector of A. What is the corresponding eigenvalue?

10. Let $A(t) = \big[a_{ij}(t)\big]$ be a square matrix whose coefficients are real and depend smoothly on a real parameter t.

a) If $\lambda(0)$ is a simple eigenvalue (that is, its algebraic multiplicity is one, so $\lambda(0)$ is a "simple" root of the characteristic polynomial), show that $\lambda(t)$ is a smooth functions of t for t sufficiently small.

b) If the above matrix $A(t)$ is self-adjoint (symmetric) with $A(0)v = \lambda(0)v$, derive the formula

$$\lambda'(0) = \frac{\langle v,\, A'(0)v\rangle}{\|v\|^2} \qquad \left(\text{here} \quad ' = \frac{d}{dt}\right).$$

Bonus Problem

[Please give these directly to Professor Kazdan]

B-1 If A is a square matrix that is sufficiently close to the identity matrix, show that it has a square root, that is, there is a matrix B with $B^2 = A$. Moreover this matrix B is unique if it is required to be near the identity matrix.

B-2 If $h(x, y) = x^2 - 2xy + 5y^2$, since then $h(x, y) = (x - y)^2 + 4y^2$, it is clear that under the change of coordinates $u = x - y$, $v = 2y$ we can write $h = u^2 + v^2$ as a sum of squares. Prove that one can do this near the origin for any smooth function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ with the properties that

$$f(0, 0) = 0, \qquad f'(0, 0) = 0, \qquad f''(0, 0) \text{ is positive definite.}$$

[Here f' is the gradient and f'' the second derivative matrix.]

B-3 [Marsden-Hoffman, p. 413 #4]

a) If the smooth function f has a non-degenerate critical point at $x_0 \in \mathbb{R}^n$, show there is a neighborhood of x_0 containing no other critical points.

b) Find the critical points of $f(x, y) = x^2y^2$.

[Last revised: October 22, 2015]