Math 361, Fall 2015 **Jerry L. Kazdan**

Problem Set 5

DUE: Thurs. Oct. 15 in class. [Late papers will be accepted (without penalty) until 1:00 PM Friday.]

No class or recitation on Tuesday, Oct. 6. or during Fall Break on Thurs. Oct. 8.

Please carefully read all of Chapter 7.1-7.4 and the corresponding Proofs in the Marsden-Hoffman text.

The following short answer problems from Marsden-Hoffman are not assigned, but you should know how to do them. Some short answer problems will be on our exams.

p. 334 #1, 3, 4, p. 344 #3, p. 349 #5, p. 352 #2, 3, p. 355 #1, 6,
p. 362 #5, p. 367 #1, 2, 3, p. 384 #4, 5(g), p. 384 #7, p. 385 #8,
p. 386 # 18, 19, p. 388 #32, p. 389 # 41

1. [p. 344 #4] Find the equation of the tangent plane to the surface $z = x^3 + y^4$ at $x = 1$, $y = 3$.

2. Say $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and the level set $f(x) = 0$ defines a smooth surface. If $P \in \mathbb{R}^3$ is a point *not* in the surface and Q is a point in the surface that is closest to P, show that the vector from P to Q is normal to the surface.

3. [p. 348 #1] Write out the chain rule for
$$h(x, y, z) = f(u(x, y, z),\, v(x, y),\, w(y, z)),$$
where f, u, v, and w are assumed to be smooth functions of their variables.

4. [p. 352 #5] Let both f and g map $\mathbb{R}^n$ to $\mathbb{R}$ be smooth functions. Show that
$$\operatorname{grad}(fg) = f \operatorname{grad} g + g \operatorname{grad} f.$$

5. [p. 386 #16] If $f : \mathbb{R}^n \to \mathbb{R}^k$ is a differentiable map and Df is a constant, show that f is an affine map (a linear term plus a constant) and that the linear part of f is the constant value of Df.

6. Find and classify the critical points of the following functions:
   a) $f(x, y) := x^3 - 3x + y^2$.
   b) $g(x, y, z) := x^3 - 3x + y^2 + (z - 1)^2$.

7. Find and classify the critical points of $f(x, y) = (x^2 + 3y^2)e^{(1-x^2-y^2)}$.

8. Find an example of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ that has exactly three critical points: one a local max, one a local min, and one a saddle point.

9. Let $f(x, y) = (y - x^2)(y - 2x^2)$.
   a) Find and classify the critical points of f. It may help to draw a sketch of the xy-plane and indicate where f is positive, negative, and zero.
   b) Show that if you approach the origin *only* along any straight line the function seems to have a local minimum at the origin. This deception is because the origin is a *degenerate critical point*, that is, the second derivate matrix (the *Hessian*) is not invertible. [Bonus Problem 1 below discusses this further].

10. SOME BASIC NONLINAR MAPS
    a) Find a smooth invertible map $y = f(x)$ from the real line to the positive reals: $0 < y < \infty$. [SUGGESTION: First draw a sketch of the graph; then "recognize" a function that looks like this.]
    b) Find a smooth invertible map f from the real line to the interval: $-1 < y < 1$.
    c) Find a smooth invertible map f from $\mathbb{R}^2$ to the right-half plane: $\{(y_1, y_2) \in \mathbb{R}^2 : y_1 > 0\}$.
    d) Find a smooth invertible map f from $\mathbb{R}^2$ to the half plane: $\{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 > 0\}$.
    e) Find a smooth invertible map f from the half plane: $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 > 0\}$ to all of $\mathbb{R}^2$.
    f) Find a smooth invertible map f from $\mathbb{R}^2$ to the open square in $\mathbb{R}^2$ with vertices at $(-1, -1)$, $(1, -1)$, $(1, 1)$, and $(-1, -1)$.

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 Say the smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ has a critical point at the origin and that this critical point is *non-degenerate*, that is, the Hessian matrix $f''(0, 0)$ is invertible. Say when you approach the origin along any straight line, on these straight lines the function has a local minimum at the origin. Show that if you have any sequence of points $(x_j, y_j) \to (0, 0)$, then f has a local minimum at the origin.

REMARK: Problem 9 above has an example showing this may be false if the origin is a degenerate critical point].

B-2 a) Let $f : \mathbb{R} \to \mathbb{R}$ have a critical point at the origin and that $f''(0) > 0$. If f does not have any other critical points, show that f has its global minimum at the origin, that is, $f(x) > f(0)$ for all $x \neq 0$.

b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ have a critical point at the origin, that f'' is positive definite there, and that f does not have any other critical points.

PROOF OR COUNTEREXAMPLE: Then f has its global minimum at the origin, that is, $f(x) > f(0)$ for all $x \neq 0$.

[Last revised: September 30, 2015]