Math 361, Fall 2015 **Jerry L. Kazdan**

Problem Set 4

DUE: Thurs. Sept. 24 in class. [Late papers will be accepted (without penalty) until 1:00 PM Monday.]

Please carefully read all of Chapter 6 in the Marsden-Hoffman text.

The following short answer problems from Marsden-Hoffman are not assigned, but you should know how to do them. Some short answer problems will be on our exams.

p. 330 #4, 5, p. 334 #2, 3, 4, p. 344 #1, 4, p. 348 #2, 5, p. 352 #3, 4, p. 384 #4

1. If the (possibly complex) sequence $\{a_n\}$ is bounded and $c > 1$, show that the series $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ converges absolutely and uniformly for all complex $z = x + iy$ in the closed half-plane $c \le x < \infty$.

2. Let $f \in C([0. 2\pi])$. Show that $\lim_{n\to\infty} \int_0^{\pi} f(x) \sin nx \, dx = 0$.

   [There are several ways to do this. One is with your bare hands. Another is to approximate f uniformly by a "simpler" function.]

3. Say you have k linear algebraic equations in n variables; in matrix form we write $AX = Y$. Give a proof or counterexample for each of the following.

   a) If $n = k$ there is always *at most one* solution.

   b) If $n > k$ you can *always* solve $AX = Y$.

   c) If $n > k$ the nullspace of A has dimension greater than zero.

   d) If $n < k$ then for *some* Y there is *no* solution of $AX = Y$.

   e) If $n < k$ the *only* solution of $AX = 0$ is $X = 0$.

4. Let L, M, and N be linear maps from the (two dimensional) plane to the plane given in terms of the standard $\mathbf{i}$, $\mathbf{j}$ basis vectors by:

   $L\mathbf{i} = \mathbf{j}$, $L\mathbf{j} = -\mathbf{i}$ (rotation by 90 degrees counterclockwise)

   $M\mathbf{i} = -\mathbf{i}$, $M\mathbf{j} = \mathbf{j}$ (reflection across the vertical axis)

   $NV = -V$ (reflection across the origin)

   a) Draw pictures describing the actions of the maps L, M, and N and the compositions: LM, ML, LN, NL, MN, and NM.

   b) Which pairs of these maps commute?

c) Which of the following identities are correct—and why?

1)	$L^2 = N$	2)	$N^2 = I$	3)	$L^4 = I$	4)	$L^5 = L$
5)	$M^2 = I$	6)	$M^3 = M$	7)	$MNM = N$	8)	$NMN = L$

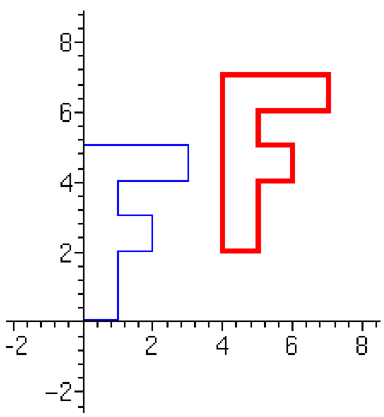
d) Find matrices representing each of the linear maps L, M, and N.

5. Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

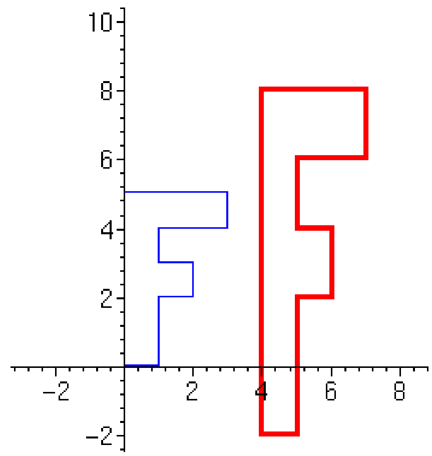
$$F(X) = V + AX,$$

where V is a vector. These are called *affine maps*. Note that $F(0) = V$.

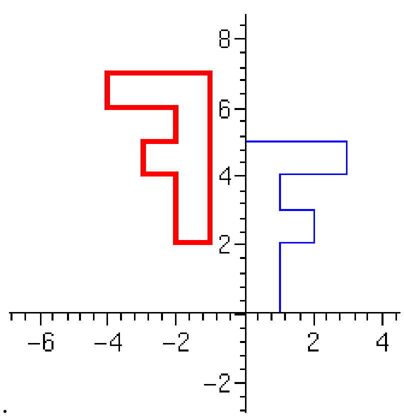
Find the vector V and the matrix A that describe each of the following mappings [here the light blue F is mapped to the dark red F].



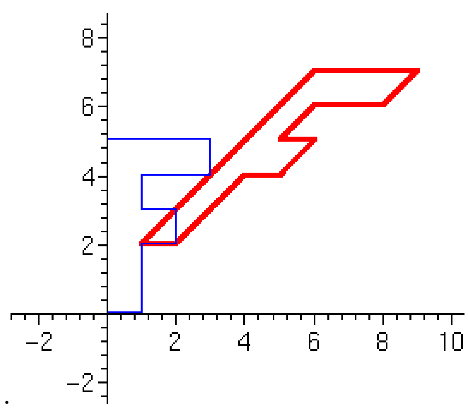
a).



b).



c).



d).

6. Let $A(t)$ be an invertible $n \times n$ matrix whose elements are smooth functions of $t \in \mathbb{R}$. Using the definition of the derivative as

$$\lim_{t \to t_0} \frac{A(t) - A(t_0)}{t - t_0},$$

show that the inverse, A^{-1}, is also differentiable and give a formula for it in terms of $A^{-1}(t)$ and $A'(t)$.

Suggestion: First do the special case $n = 1$. Also, don't assume that $A(t)$ and $A(t_0)$ commute. Key step: for invertible matrices A, B, verify and use

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}.$$

7. Let $f(t) : \mathbb{R} \to \mathbb{R}$ be any smooth function. For any real numbers a and b let $u(x, y) = f(ax + by)$. Show that

a) $bu_x(x, y) - au_y(x, y) = 0$.

b) $u_{xx}u_{yy} - (u_{xy})^2 = 0$.

8. (Euler) [Marsden-Hoffman, p. 385 #12] Let $f(x)$ be a differentiable function of $x \in \mathbb{R}^n$. If f is *homogeneous of degree* k in the sense that $f(cx) = c^k f(x)$ for all $c > 0$, show that $x \cdot \nabla f(x) = kf(x)$.

9. Let $p > 1$ and define q by $1/p + 1/q = 1$. Prove **Young's inequality**: for all real $x \geq 0$ and $y \geq 0$:

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

[Remark: There are many (elementary) ways to show this; none of them are completely obvious. In one approach, fix $y = a > 0$ and find the maximum of $g(x) := ax - x^p/p$].

10. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and assume that $u(x)$ depends only on the distance to the origin, $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, so $u(x) = f(r)$ for some function f depending only on r.

a) Show that $\dfrac{\partial u}{\partial x_i} = \dfrac{x_i}{r}\dfrac{df}{dr}$.

b) Show that $\dfrac{\partial^2 u}{\partial x_i^2} = \dfrac{d^2 f}{dr^2}\left(\dfrac{x_i^2}{r^2}\right) + \dfrac{df}{dr}\left(\dfrac{1}{r} - \dfrac{x_i^2}{r^3}\right)$.

c) Compute $\Delta u := \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \ldots + \dfrac{\partial^2 u}{\partial x_n^2}$ in terms of f and its derivatives.

d) If $n = 3$, use this to find all functions $u(x) = f(r)$ that satisfy $\Delta u = 0$ for all $x \neq 0$.

[Last revised: September 25, 2015]