Math 361, Fall 2015 **Jerry L. Kazdan**

Problem Set 3

DUE: Thurs. Sept. 17 in class. [Late papers will be accepted (without penalty) until 1:00 PM Friday.]

Please carefully read Sections 6.1–6.5 in the Marsden-Hoffman text.

The following short answer problems from Marsden-Hoffman are not assigned, but you should know how to do them. Some short answer problems will be on our exams.

p. 244 #3, p. 272 #1, p. 272 #2, p. 282 #4, p. 283 #8,
p. 317 #7, p. 317 #8, p. 317 #18, p. 320 #29(b,c, d), p. 320 #32

1. [Parts of p. 317#10]. For each of the sequences a)-d) in #2, when can we integrate or differentiate term-by-term? Justify your response.

2. [Marsden-Hoffman, p. 320 #34]

3. Let X and Y be linear spaces and $L : X \to Y$ be a linear map.

   Say x_1 and x_2 are particular solutions of the equations $Lx = y_1$ and $Lx = y_2$, respectively, while $z \neq 0$ is a solution of the homogeneous equation $Lz = 0$. Answer the following in terms of x_1, x_2, and z.

   a) Find some solution of $Lx = 3y_1$.
   b) Find some solution of $Lx = -5y_2$.
   c) Find some solution of $Lx = 3y_1 - 5y_2$.
   d) Find another solution (other than z and 0) of the homogeneous equation $Lx = 0$.
   e) Find *two* solutions of $Lx = y_1$.
   f) Find another solution of $Lx = 3y_1 - 5y_2$.

4. In class, we showed that iff $f(x)$ and $K(x, y)$ are continuous function of x and y for x and y in the interval $0 \le x \le 1$, AND if $|\lambda|$ is sufficiently small, then the integral equation

$$u(x) = f(x) + \lambda \int_0^1 K(x, y)u(y)\, dy$$

   has a unique solution $u \in C([0, 1])$.

   By looking at the special case where $K(x, y) \equiv 1$, and $f(x) \equiv 2$, show that the assumption that λ is sufficiently small cannot be eliminated completely.

5. For each of the following give an example of a sequence of continuous functions. If you prefer, a clear sketch of a graph will be adequate.

   a) $f_n(x) \to 0$ for all $x \in [0,\,1]$ but $\displaystyle\int_0^1 f_n(x)\,dx \ge 1$ for all $n = 1, 2, \ldots$.

   b) $g_n(x) \to 0$ for all $x \in [0,\,1]$ and $\displaystyle\int_0^1 g_n(x)\,dx \to 0$ but the g_n do not converge uniformly to zero on $[0,\,1]$.

6. Suppose that $G : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function with the property that for some real M

$$\|G(x) - G(y)\| \le M\|x - y\| \qquad \text{for all } \; x, y \in \mathbb{R}^n. \tag{1}$$

   Here $\|x\|$ is the standard Euclidean distance in $\mathbb{R}^n$.

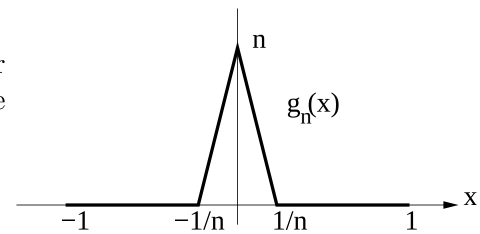
   If $\lambda > 0$ is small enough, show that the function $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F(x) = x - \lambda G(x)$$

   is one-to-one and onto, so for every $z \in \mathbb{R}^n$ the equation $F(x) = z$ has one and only one solution $x \in \mathbb{R}^n$. Note that a solution x is a fixed point of some map.

7. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be continuous for all x with $f(x) = 0$ for $|x| \ge 1$ and let $g_n(x)$ be the sequence of functions in the figure. Let

$$h_n(t) = \int_{-1}^{1} f(t - x) g_n(x)\,dx$$



   a) Show that h_n is uniformly continuous.

   b) Show that $\displaystyle\lim_{n\to\infty} h_n(t) = f(t)$ uniformly.

[Last revised: September 11, 2015]