Math 361, Fall 2015 **Jerry L. Kazdan**

Problem Set 2

DUE: Thurs. Sept. 10 in class. [Late papers will be accepted (without penalty) until 1:00 PM Friday.]

Please carefully read Sections 5.1–5.5 and 5.7 in the Marsden-Hoffman text.

1. Prove the *Integral Intermediate Value Theorem*: If f is real and continuous on $[a, b]$, then there exists $c \in (a, b)$ such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c).$$

   Also, give an example showing that such a c may not exist if f is not continuous.

2. [Like Marsden-Hoffman p. 316 #2] Determine which of the following function sequences of functions converge pointwise or uniformly:

   a) $f_n(x) = \frac{\sin x}{n}$, $x \in \mathbb{R}$

   b) $f_n(x) = \frac{1}{1+nx}$, $x \in [0,\, 1]$

   c) $f_n(x) = \frac{x}{1+nx^2}$, $x \in \mathbb{R}$.

3. [Marsden-Hoffman p. 245 Ex. 4] Discuss the uniform convergence of $\sum_{n=1}^{\infty} 1/(x^2+n^2)$.

4. A continuous function is called *piecewise linear* if it consists only of straight line segments (see `https://en.wikipedia.org/wiki/Piecewise_linear_function`)

   Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that given any $\epsilon > 0$, there is a piecewise linear function $g : [a, b] \to \mathbb{R}$ such that $|f(x) - g(x)| < \epsilon$ for all $x \in [a, b]$. In other words, any continuous function on $[a, b]$ can be approximated "uniformly" by a piecewise linear function.

5. Give a proof or a counterexample for the following questsions.

   a) If $f_n \to f$ and $g_n \to g$ pointwise in $C([0,\, 1])$. is it true that $f_n \cdot g_n \to f \cdot g$ pointwise?

   b) If $f_n \to f$ and $g_n \to g$ uniformly in $C([0,\, 1])$, is it true that $f_n \cdot g_n \to f \cdot g$ uniformly?

6. Let $f \in C([0, \infty))$ be a continuous function with the property that $\lim_{x\to\infty} f(x) = c$. Show that

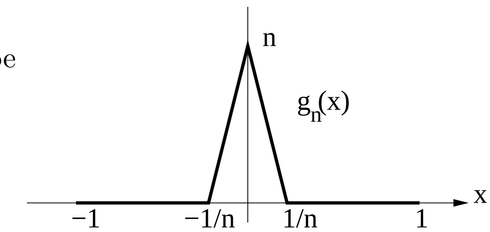
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\,dx = c.$$

7. Compute $\lim_{\lambda\to\infty} \int_0^1 |\sin(\lambda x)|\, dx$.

8. Let $f(x)$ be continuous on the interval $[-1, 1]$ and $g_n(x)$ be the sequence of functions in the figure. Show that

$$\lim_{n\to\infty} \int_{-1}^1 f(x) g_n(x)\, dx = f(0).$$

SUGGESTION First do the case where $f(x) \equiv 1$.



Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 Let $f(x)$ be a continuous function for $0 \le x \le 1$. Evaluate $\lim_{n\to\infty} \int_0^1 n f(x) x^n\, dx$. (Justify your assertions.)

[Last revised: September 4, 2015]