**Math 361, Fall 2015** **Jerry L. Kazdan**

# Problem Set 10

Due: Thurs. Dec. 3 in class. [Late papers will be accepted (without penalty) until 1:00 PM Friday.]

Please carefully read Chapter 9 in the Marsden-Hoffman text.
The following short answer problems from Marsden-Hoffman are not assigned, but you should know how to do them. Some short answer problems will be on our exams.

p. 508 #5, p. 512 #4, p. 539 #24

1. Compute the arc length of the spiral helix $\gamma(t) = (t\cos t,\, t\sin t, t)$ for $0 \le t \le 2\pi$.

2. Consider the curve $y = x\sin(1/x)$ for $0 < x \le 1$ while $y = 0$ when $x = 0$. Show that this curve is not rectifiable (has infinite length).

3. Let $A$ be a real positive definite $n \times n$ matrix and $c > 0$. Compute the "volume" of the set of points in $x \in \mathbb{R}^n$ where $\langle x,\, Ax\rangle \le c$.

4. Let $A$ be a real positive definite $n \times n$ matrix, $b$ a vector in $\mathbb{R}^n$ and $c$ a real number. Fow which value(s) of $c$ is the set of points in $x \in \mathbb{R}^n$ where
$$\langle x,\, Ax\rangle + 2\langle b,\, x\rangle + c = 0$$
not empty? [Suggestion: First answer this in the special case $n = 1$.].

5. Let $\gamma(s)$ define a smooth curve parametrized by arc length $s$, so $\|\gamma'(s)\| = 1$. Show that the vector $\gamma''(s)$ is perpendicular to the tangent vector $\gamma'(s)$.

6. Let $0 < b < a$. In class we parametrized the standard torus (surface of a doughnut) in $\mathbb{R}^3$ as
$$x = (a + b\cos\phi)\cos\theta, \qquad y = (a + b\cos\phi)\sin\theta, \qquad z = b\sin\phi.$$
Show this is a smooth two dimensional surface and compute the equation of the tangent plane at the point $\theta = \pi/2$, $\phi = 0$.

## Bonus Problems

[Please give these directly to Professor Kazdan]

B-1 a) Let $V = (v_1, \dots, v_n) \neq 0$ be a real vector and $A$ be the $n \times n$ matrix $A = (v_i v_j)$, so the product $v_i v_j$ is the $ij$ element of $A$. Find the eigenvalues of $A$. [Note: $A$ has rank one since each column is a multiple of $V$. Thus $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$. What is the trace of $A$? What is $\lambda_n$?]

Let $B = A + cI$, where $c \in \mathbb{R}$. Compute the eigenvalues and the determinant of $B$.

b) If $u(x_1, x_2, \dots, x_n)$ is a given smooth function, let $u'' := \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ be its second derivative (Hessian) matrix. Find all solutions of $\det(u'') = 1$ in the special case where $u = u(r)$ depends only on $r = \sqrt{x_1^2 + \cdots x_n^2}$, the distance to the origin.

c) Let $x = (x_1, x_2, \dots, x_n)$ and $A$ be a square matrix with $\det A = 1$. If $u(x)$ satisfies $\det(u'') = 1$ (see above), and $v(x) := u(Ax)$, show that $\det(v'') = 1$ also. [Remark: the differential operator $\det(u'')$ is interesting because its symmetry group is so large.]

B-2 a) On the real number line, consider two points $x$ and $\hat{x}$ to be equivalent if $x - \hat{x} = k$ for an integer $k$. If $c$ is an irrational number, show that the points $nc$, $n = 1, 2, \dots$ are all distinct and are dense in the interval $[0, 1]$.

b) The *flat torus* in $\mathbb{R}^2$ is the set of points $x$, $y$ where we consider the point $\hat{x}$, $\hat{y}$ to be equivalent to $(x,\ y)$ if $x - \hat{x}$ and $y - \hat{y}$ differ by integers. (possibly different integers). Consider the line $y = cx$ in the plane where $c$ is an irrational number. Show that viewing this line as on the flat torus, it does not intersect itself and its points are dense in the torus. [Suggestion: First understand the simpler case where $c$ is a rational number and show that you get a closed curve.]

[Last revised: December 3, 2015]