Problem Set 1

DUE: Thurs. Sept. 3 in class. [Late papers will be accepted (without penalty) until 1:00 PM Friday.]

Please review all of Chapters 1–5 in the Marsden text.

1. Let $\{\vec{v}_j\}$ and $\{\vec{w}_j\}$ be sequences of points in $\mathbb{R}^n$ and let $\vec{v}_j \cdot \vec{w}_j$ be their inner ("dot") product. If $\vec{v}_j \to \vec{v}$ and $\vec{w}_j \to \vec{w}$, show that $\vec{v}_j \cdot \vec{w}_j \to \vec{v} \cdot \vec{w}$.

2. Let $f : [0, 2] \to [0,\ 2]$. be a continuous function.
   a) Show that there is at least one point $c \in [0,\ 2]$ so that $f(c) = c$.
   b) Give an example showing that there may be more than one such point.
   c) Say $|f'(x)| < 1$ for all $x \in [0,\ 2]$. Show there is exactly one such point,

3. a) Show that at any time, there are always at least two antipodal points on the equator with the same temperature.
   b) Generalize.

4. a) Let $\{a_j\} \in \mathbb{R}$ be a sequence of real numbers with the property that
   $$|a_{j+1} - a_j| \le \tfrac{1}{2}|a_j - a_{j-1}|.$$
   Show that the $\{a_j\}$ are a Cauchy sequence and hence converge to some real number A.
   HINT: If $n > k$, then $\quad a_n - a_k = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{k+1} - a_k)$.
   b) Let $\{\vec{v}_j\} \in \mathbb{R}^n$ be a sequence of points with the property that
   $$\|\vec{v}_{j+1} - \vec{v}_j\| \le \tfrac{1}{2}\|\vec{v}_j - \vec{v}_{j-1}\|.$$
   Show that the $\{\vec{v}_j\}$ are a Cauchy sequence and hence converge to some point $p \in \mathbb{R}^n$.

5. Find a continuous function f and a constant C so that
   $$\int_0^x f(t)\,dt = e^{\sin x} + C.$$

6. Let $f : (0, \infty) \to (0, \infty)$, be a continuous and decreasing function. Prove that the sequence
   $$S_n := f(1) + \cdots + f(n) - \int_1^{n+1} f(t)\,dt$$
   is convergent and that its limit lies in the closed interval $[0, f(1)]$.

7. Let $f := \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$|f'(x)| \leq 2015 \qquad \text{for all } x \in \mathbb{R}.$$

   Prove that f is uniformly continuous on $\mathbb{R}$.

8. Let $f(x)$ be a twice continuously differentiable function with the property that $f''(x) \geq 0$ for all $x \in \mathbb{R}$. If f is bounded from above show that $f(x)$ must be a constant.

9. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a differentiable function whose derivative g' satisfies the inequality $|g'(x)| < M$ for all $x \in \mathbb{R}$.

   Show that if $\varepsilon > 0$ is small enough, then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + \varepsilon g(x)$ is one-to-one and onto, so for every $y \in \mathbb{R}$ the equation $f(x) = y$ has one and only one solution.

[Last revised: August 30, 2015]