Math 361, Fall 2015 **Jerry L. Kazdan**

Problem Set 0 [Rust Remover]

DRAFT

DUE: Never – but you will be expected to know this material.

This week: Please review all of Chapters 1–5 in the Marsden text.

1. Let $a_n > 1$ be a sequence of real numbers. If $a_n \to A$, show that $A \geq 1$.

2. Let $a_n \neq 0$ be a sequence of real numbers. If $a_n \to A$ and $A \neq 0$, show that $1/a_n$ converges to $1/A$.

3. In $\mathbb{R}$, if $a_n \to A$ and $b_n \to B$, show that the product $a_n b_n \to AB$.

   If instead you assume that $a_n \to 0$ and b_n is bounded, does $a_n b_n \to 0$? Proof or counterexample.

4. Let $\{\vec{v}_j\}$ and $\{\vec{w}_j\}$ be sequences of points in $\mathbb{R}^3$ and let $\vec{v}_j \cdot \vec{w}_j$ be their inner ("dot") product. If $\vec{v}_j \to \vec{v}$ and $\vec{w}_j \to \vec{w}$, show that $\vec{v}_j \cdot \vec{w}_j \to \vec{v} \cdot \vec{w}$.

5. Let $\vec{v}$ and $\vec{w}$ be points in the plane, $\mathbb{R}^2$. Show that
$$\|\vec{v} + \vec{w}\|^2 - \|\vec{v} - \vec{w}\|^2 = 4\vec{v} \cdot \vec{w}.$$

6. Let $f : [0, 2] \to [0,\ 2]$. be a continuous function.
   a) Show that there is at least one point $c \in [0,\ 2]$ so that $f(c) = c$.
   b) Give an example showing that there may be more than one such point.
   c) If in addition $f(x)$ is differentiable and $|f'(x)| < 1$ for all $x \in [0,\ 2]$, show there is exactly one such point,

7. EXAMPLES:
   a) Give an example of a bounded set of real numbers that has exactly two limit points.
   b) Give an example of a bounded subset of $\mathbb{R}^2$ that contains all but one of its limit points.
   c) Give an example of an open cover of $\{x \in \mathbb{R} \ : \ 0 < x \leq 1\}$ that has no finite sub-cover.

8. Let $f : \{-1 < x < 2\} \to \mathbb{R}$ be a continuous function. If $f(0) = 6$, show there is some interval $J := [-c,\ c] \subset \{-1 < x < 2\}$ with $c > 0$ so that $f(x) > 3$ for all $x \in J$.

9. [Classify sets] For each of the sets below, determine which of the following properties it has: *open* *closed* *bounded* *compact* *countable*

a) $\{x \in \mathbb{R} : x = 1, 2, 3, 4\}$

b) $\{1 - \frac{1}{n} \in \mathbb{R},\ n = 1, 2, 3, \dots\}$

c) $\{1\} \cup \{1 + \frac{(-1)^n}{n} \in \mathbb{R},\ n = 1, 2, 3, \dots\}$

d) $\{x_n = (-1)^n + \dfrac{1}{n} \in \mathbb{R}, n = 1, 2, 3, \dots\}$

e) $\{(x, y) \in \mathbb{R}^2 : x \text{ and } y \text{ are positive integers}\}$

f) $\{x \in \mathbb{R} : 0 < x < 1\}$

g) $\{(x,\ y) \in \mathbb{R}^2 : 0 < x < 1,\ y = 0\}$

h) $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$

i) $\{x \in \mathbb{R} : x = 1, 1/2, 1/3, \dots\}$

j) $\{x \in \mathbb{R} : x = 1, 1/2, 1/3, \dots\} \cup \{0\}$

k) $\{(x, y) \in \mathbb{R}^2 : 0 \leq y - x \leq 1\}$

l) $\{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2\}$

m) $\{(x, y) \in \mathbb{R}^2 : x > 1,\ y < 1/x\}$

n) $\{x \in \mathbb{R} : x \text{ is a rational number}\}$

o) $\{(x, y) \in \mathbb{R}^2 : x \text{ and } y \text{ are rational numbers}\}$

p) $\{(k, n) \in \mathbb{R}^2 : k, n \text{ any positive integers with } k^2 + n^2 < 100\}$

q) $\{(x, y) \in \mathbb{R}^2 : x + y > 1\}$

r) $\{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 4\}$

s) The following points in $\mathbb{R}^4$: $\{e_1 = (1,\ 0,\ 0,\ 0),\quad e_2 = (0,\ 1,\ 0,\ 0),$
$e_3 = (0,\ 0,\ 1,\ 0,\ ),\quad e_4 = (0,\ 0,\ 0,\ 1)\}$

10. Let J be the interval $J := \{0 \leq x \leq 1\}$.

a) Let $f(x) = 0$ for all $x \in J$ *except* that $f(1/2) = 1$. Show that f is Riemann integrable on J and compute the value of the integral.

b) Say $f(x)$ is continuous on J and $f(x) \geq 0$ there. If $\int_0^1 f(x)\, dx = 0$, show that $f(x) = 0$ in J.

c) Say $f(x)$ is continuous in J and $\int_0^1 f(x)g(x)\, dx = 0$ for *every* function $g(x)$ that is continuous in J. Show that $f(x) = 0$ at every point of J.

11. a) Let $a_j \in \mathbb{R}$ be a sequence of real numbers with the property that

$$|a_{j+1} - a_j| \leq \tfrac{1}{2}|a_j - a_{j-1}|.$$

Show that $|a_{j+1} - a_j| \leq 2^{-j}|a_1 - a_0|$.

Use this to show that a_j is a Cauchy sequence and hence converge to some real number A.

HINT: If $n > k$, then $\quad a_n - a_k = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{k+1} - a_k)$.

b) Let $\vec{v}_j \in \mathbb{R}^n$ be a sequence of points with the property that

$$\|\vec{v}_{j+1} - \vec{v}_j\| \le \tfrac{1}{2}\|\vec{v}_j - \vec{v}_{j-1}\|.$$

Show that $\vec{v}_j$ is a Cauchy sequence and hence converge to some point $p \in \mathbb{R}^n$.

[Last revised: June 21, 2015]