

Analysis Problems

Penn Math

Jerry L. Kazdan

In the following, when we say a function is smooth, we mean that all of its derivatives exist and are continuous.

These problems have been crudely sorted by topic – but this should not be taken seriously since many problems fit in a variety of topics.

1. A straight line ℓ is tangent to the cubic $y = x^3 + bx^2 + Cx + d$. Let Q be the bounded region between the line and the cubic. Let ℓ' be the (unique) line that is parallel to ℓ and is also tangent to the same cubic. This also defines a bounded region, R , between ℓ' and the cubic.

- a) Show that Q and R have the same areas.
- b) What else can you say about the relationship between Q and R ?

2. Find the elementary function having the following power series for $|z| < 1$.

a) $\sum_{n=1}^{\infty} nz^n$ b) $\sum_{n=0}^{\infty} \frac{z^n}{n+1}$

3. Determine the sum of each of the following numerical series.

a) $\sum_{n=0}^{\infty} \frac{1}{2^n n!}$ b) $\sum_{n=1}^{\infty} \frac{1}{n 2^n}$ c) $\sum_{n=0}^{\infty} \frac{2n+1}{2^n}$ d) $\sum_{n=0}^{\infty} \frac{(n+1)^2}{2^n}$ e) $\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$

4. Compute $\lim_{n \rightarrow \infty} \left[(n+1)^{2/7} - n^{2/7} \right]$.

5. Let a_1, a_2, \dots, a_n be a sequence of complex numbers and let S_n be their arithmetic mean

$$S_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

- a) If a_n converges to A , show that S_n also converges to A .
- b) Give an example where the S_n converges but the a_n do not converge.
- c) If the S_n converge, is the sequence a_n necessarily bounded? Proof or counterexample.

d) If the a_n are monotonic increasing and unbounded, is the sequence S_n also unbounded? Proof or counterexample.

6. Let $f(x)$ be a continuous real-valued function. If $\lim_{x \rightarrow \infty} f(x) = C$, show that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x) dx = C.$$

7. Let $b_1 \leq b_2 \leq \dots$ be a monotonic increasing sequence of real numbers and let

$$T_n = b_1 + b_2 + \dots + b_n.$$

a) For any integer $n \geq 1$ show that $nb_n \leq nb_{n+1} \leq T_{2n} - T_n \leq nb_{2n}$.

b) Show that

$$b_1 + 2b_2 + 4b_4 + 8b_8 + \dots + 2^n b_{2^n} \leq T_{2^{n+1}} - T_1 \leq b_2 + 2b_4 + 4b_8 + 8b_{16} + \dots + 2^n b_{2^{n+1}}.$$

c) Apply this to estimate $1^p + 2^p + \dots + n^p$ for real $p > 1$.

8. Determine if the sequence $\frac{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}}{n}$ converges or diverges.

9. a) Find a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f^{(n)}(0) = n$ for all $n \geq 0$. Your answer should be expressed in terms of the elementary functions encountered in calculus (such as x , $\sin x$, e^x , etc.).

b) The same question with $f^{(n)}(0) = n^2$, for each n .

c) Let $g(x)$ be a smooth function and say $g^{(n)}(0) = b_n$. Find a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f^{(n)}(0) = nb_n$ for all $n \geq 0$.

10. Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that for all x , $0 \leq x \leq 2$ you know that $f'(x) = 0$. Prove that f is constant for $0 \leq x \leq 2$. Include very brief self-contained proofs of *all* the preliminary results you use (for instance, that a continuous function on a closed and bounded interval attains its maximum at some point on the interval).

11. Let $f(x)$ be a continuous real-valued function with the property

$$f(x+y) = f(x) + f(y)$$

for all real x, y . Show that $f(x) = cx$ where $c := f(1)$. [REMARK: There is a very short proof if you assume f is differentiable].

12. Assume that f is defined in a neighborhood U of $x_0 \in \mathbb{R}$ and has derivatives up to order $2n - 1$ for all $x \in U$ with $f'(x_0) = f''(x_0) = \dots = f^{(2n-1)}(x_0) = 0$. Assume that $f^{(2n)}(x_0)$ exists and is positive. Show that f has a local minimum at x_0 .

13. Let $\{b_n\}$, $n = 1, 2, \dots$ be a sequence of real numbers with the property that $b_{n+k} \leq b_n + b_k$. Prove that $\lim_{n \rightarrow \infty} \frac{b_n}{n}$ exists.

14. Consider the sequence:

$$x_{n+1} = a^{x_n}$$

Does it converge for $a = \sqrt{2}$? What about other values of $a > 0$? [If it converges, its limit is the intersection of $y = a^x$ with $y = x$.]

15. Let $\{a_n\}_{n=1}^{\infty}$ be a *bounded* sequence of real numbers with the property that $|a_n - a_{n-1}| < \frac{1}{n}$ for all $n \geq 1$. Must it converge? Proof or counterexample.

16. a) Let $X_j, j = 1, 2, \dots$ be a sequence of points in \mathbb{R}^3 . If $\|X_{j+1} - X_j\| \leq \frac{1}{j^4}$, show that these points converge.

b) Let $\{X_j\}$ be a sequence of points in \mathbb{R}^n with the property that

$$\sum_j \|X_{j+1} - X_j\| < \infty.$$

Prove that the sequence $\{X_j\}$ converges. Give an example of a convergent sequence that does not have this property.

17. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with $|f(x) - f(y)| \leq c|x - y|$ for some $c < 1$ and all x, y . Show that f has *fixed point* (this is a point α with $f(\alpha) = \alpha$) and that this fixed point is unique.

[SUGGESTION: First show that there is at most one fixed point. Then, for any x_0 , show that the sequence x_k defined recursively by $x_k = f(x_{k-1})$, $k = 1, 2, \dots$, converges to the desired fixed point.]

Give an example with $c = 1$ where f does not have a fixed point.

18. Let $S = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$.

a) Give an example of a function $f : S \rightarrow \mathbb{R}$ with the property that for some c we have $|f(x) - f(y)| \leq c|x - y|^{1/3}$ for all x, y in S , but f is not everywhere differentiable.

- b) If for some $\alpha > 1$ and some $c > 0$ a function $h : S \rightarrow \mathbb{R}$ satisfies $|h(x) - h(y)| \leq c|x - y|^\alpha$ for all x, y in S , what can you conclude about h ?
19. a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function with the properties that $f''(x) \geq 0$ and $f(x) \leq C$ for all $x \in \mathbb{R}$. Show that $f(x) = \text{constant}$.
- b) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth function with the properties that the hessian matrix $f''(x)$ is positive semi-definite and that $f(x) \leq C$ for all $x \in \mathbb{R}^2$. Does this imply that $f(x) = \text{constant}$? Proof or counterexample.
20. Let C be the ring of continuous functions on the interval $0 \leq x \leq 1$.
- a) If $0 \leq c \leq 1$, show that the subset $\{f \in C \mid f(c) = 0\}$ is a maximal ideal.
- b) Show that every maximal ideal in C has this form.
21. Every maximal ideal of $C([0, 1])$ is prime. Proof or counterexample.
22. Let $f(x) : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded smooth function for $-\infty < x < \infty$. Show that $f''(c) = 0$ for at least one point c . Thus, f has at least one inflection point.
23. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that: for every $a, b \in \mathbb{R}$ with $a < b$, there is a *unique* $\xi \in (a, b)$ such that $f'(\xi) = \frac{f(a) - f(b)}{a - b}$. Prove that f has no inflection points.
24. Show that $\lim_{x \rightarrow \infty} \frac{f(x + \sin x)}{x} = \lim_{x \rightarrow \infty} f'(x)$, provided the limit on the right hand side exists and is finite.
25. *Newton's method* for solving $f(x) = 0$ given the approximate guess x_k tells us to find the next approximation, x_{k+1} , by solving $f(x_k) + f'(x_k)(x_{k+1} - x_k) = 0$ for x_{k+1} . These approximations may or may not converge, depending both on the function $f(x)$ and how close x_k is to the desired solution.
- Show that if $f(x) = x^2 - c$, where $c > 0$, and if $x_0 > 0$, then Newton's method always converges to compute \sqrt{c} .
26. Show, directly from the definition, that \sqrt{x} is continuous at every $x \geq 0$. Is it uniformly continuous for every $x \in [0, \infty)$? Why?

27. Which of the following are uniformly continuous in the set $\{x \geq 0\}$? Justify your assertions.
- a). $f(x) = 2 + 3x$ b). $g(x) = \sin 2x$ c). $h(x) = x^2$ d). $k(x) = \sqrt{x+1}$,
28. Assume that $f(x)$ is uniformly continuous on the bounded open interval $a < x < b$. Prove that f is bounded, that is, there is some M so that $|f(x)| \leq M$ for all $x \in (a, b)$.
29. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly continuous. Prove that there are constants a, b such that $|f(x)| \leq a + b|x|$ for all x .
30. Let a function $g(x)$ be defined for $|x| \leq 1$.
- a) Find an even function, $g_{\text{even}}(x)$, and an odd function, $g_{\text{odd}}(x)$, so that $g(x) = g_{\text{even}}(x) + g_{\text{odd}}(x)$.
- b) Let $f(x)$ be an even function with the property that $|f(x) - g(x)| < \varepsilon$ for $|x| \leq 1$. Show that $|f(x) - g_{\text{even}}(x)| < \varepsilon$.
- c) Say a polynomial $p(x)$ has the property that $|f(x) - p(x)| < \varepsilon$ for $|x| \leq 1$. If $f(x)$ happens to be even, find an even polynomial with the same property.
31. Let $a < b < c$ be real numbers. Find a continuous invertible map $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ with the given following properties — or prove there is no such map.
- a) $\varphi(0) = a, \quad \varphi(1) = b, \quad \varphi(2) = c,$
- b) $\varphi(0) = c, \quad \varphi(1) = a, \quad \varphi(2) = b,$
- c) $\varphi(0) = c, \quad \varphi(1) = b, \quad \varphi(2) = a.$
32. Find diffeomorphisms (i.e., smooth invertible maps) between the following sets:
- a) $\mathbb{R} \rightarrow (0, \infty)$
- b) $\mathbb{R} \rightarrow (0, \infty)$
- c) $\mathbb{R} \rightarrow (-1, 1)$
- d) $(-1, 1) \rightarrow (0, \infty)$
- e) $\mathbb{R}^2 \rightarrow \{(x, y) \mid |y| > 0\}$ (upper half-plane)
- f) $\mathbb{R}^2 \rightarrow \{(x, y) \mid -1 < y < 1\}$ (horizontal strip)
- g) $\{(x, y) \mid -1 < y < 1\}$ to $\mathbb{R}^2 \rightarrow \{(u, v) \mid |v| > 0\}$
- h) Let $h : \mathbb{R} \rightarrow \mathbb{R}^+$ be a given smooth function. Find a diffeomorphism from $\{(x, y) \mid |y| > h(x)\}$ to $\{(u, v) \mid |v| > 0\}$

33. a) Let $A(t) = [a_i(t)]$ be a square matrix whose coefficients depend smoothly on a real parameter t . If $\lambda(0)$ is a simple eigenvalue (that is, its algebraic multiplicity is one), show that $\lambda(t)$ and the corresponding eigenvector $v(t)$ are smooth functions of t for t sufficiently small.

b) If the above matrix $A(t)$ is self-adjoint, derive the useful formula

$$\lambda' = \frac{\langle v, A'v \rangle}{\|v\|^2} \quad (\text{here } ' = \frac{d}{dt}).$$

34. Give an example of a 2×2 real symmetric matrix $A(t)$ with $A(0) = I$ whose elements depend smoothly on the parameter t but where the eigenvectors are *not* smooth functions of t near $t = 0$.

35. Let $A(t)$ be an invertible matrix whose elements $a_{ij}(t)$ depend smoothly on a parameter t .

a) Compute the derivative of $A^{-1}(t)$. Your formula should be a generalization of the 1×1 case where you are computing $\frac{d}{dt} \frac{1}{a(t)} = -\frac{a'}{a^2}$.

b) Compute the derivative of $\det A(t)$. The simplest special case is when $A(0) = I$ and one computes the derivative at $t = 0$.

REMARK: This problem does not require that A be invertible, but if it is, one useful way to write the answer is

$$(\det A(t))' = \det[A(t)] \operatorname{trace} [A^{-1}(t)A'(t)].$$

36. Consider the function $f(A) := (\det A)^p$ defined on the “cone” of real symmetric positive definite matrices A . For which real value(s) of p is this function convex? [SUGGESTION: The formula (problem 35b) for the derivative of the determinant will help.]

37. Let $f(x)$ be a continuous real-valued function with period 2π , so $f(x+2\pi) = f(x)$ for all real x . If also for some *irrational* $\alpha \in \mathbb{R}$ we know that $f(x+2\pi\alpha) = f(x)$ for all real x , show that $f(x) \equiv \text{constant}$. [SUGGESTION: Fourier series].

38. Find a smooth function $f : \mathbb{R} \rightarrow [0, 1]$ with the following properties:

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0; \\ f'(x) > 0, & \text{if } 0 < x < 1; \\ 1, & \text{if } x \geq 1. \end{cases}$$

39. a) Find a function $g \in C^1([0, 1])$ with the properties:

$$g(0) = 0, \quad g'(0) = 0, \quad g(1) = 1, \quad \text{and} \quad g'(1) = 1.$$

b) Find a $h \in C^\infty(\mathbb{R})$ so that $h(x) = 0$ for $x \leq 0$ and $h(x) = x$ for $x \geq 1$. [SUGGESTION: One approach uses a function $\varphi \in C^\infty(\mathbb{R})$ so that $\varphi(x) = 0$ for $x \leq 0$ and $\varphi(x) = 1$ for $x \geq 1$.]

40. a) Let $x > 0$. Show that $\lim_{x \rightarrow 0} \frac{1}{x^k} e^{-1/x} = 0$ for any integer k .

b) Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

Show that f is smooth, that is, $f(x)$ and all of its derivatives exist and are continuous for all real x . Sketch the graph.

c) Show that each of the following are smooth and sketch their graphs:

$$\begin{aligned} g(x) &= f(x)f(1-x), & h(x) &= \frac{f(x)}{f(x) + f(1-x)}, \\ k(x) &= h(x)h(4-x), & K(x) &= k(x+2), & H(x) &= K(4x) \\ G(x) &= \sum_{j=-\infty}^{\infty} g(x - \frac{j}{2}), & \Psi_n(x) &= \frac{g(x - \frac{n}{2})}{G(x)}, \\ Q(x) &= \sum_{n=-\infty}^{\infty} \Psi_n(x), & \varphi(x, y) &= K(x)K(y), \quad (x, y) \in \mathbb{R}^2 \\ \Phi(x) &= K(\|x\|), & x &= (x_1, \dots, x_n) \in \mathbb{R}^n \end{aligned}$$

41. Let $B \subset \mathbb{R}^n$ be the open unit ball $|x - a| < r$. Exhibit a smooth function $\varphi \in C^\infty(\mathbb{R}^n)$ that is positive on B and zero elsewhere. [SUGGESTION: One way to start is to use $\eta(t) = e^{-1/t}$ for $t > 0$.]

42. Let $0 < a_1 \leq b_1$ be real numbers and write $a_2 = \sqrt{a_1 b_1}$ and $b_2 = \frac{1}{2}(a_1 + b_1)$.

a) Show that $a_2 \leq b_2$.

b) Continue inductively to get a_3 and b_3 from $a_2 \leq b_2$ as above, etc. Show that $\lim a_n$ and $\lim b_n$ exist.

c) Furthermore, show that these limits are equal.

43. If we identify the points $x = 0$ and $x = 1$, we can think of the real interval $[0, 1]$ as a circle, S^1 . Thus for any real numbers x, y , we say that $x = y$ if $x - y$ is an integer. Let α be an irrational number,
- Show that the points $x_k = k\alpha$, $k = 1, 2, \dots$ are dense in this circle. [REMARK: An even stronger fact is true – the points x_k are *equidistributed*. See problem 155.]
 - Let $f : S^1 \rightarrow S^1$ be a continuous function, so $f(1) = f(0)$. If we also know that $f(x + \alpha) = f(x)$ for all $x \in S^1$, show that $f \equiv \text{constant}$.

44. a) Let c_n be a sequence of real numbers that converges to c . Show that their “average” (arithmetic mean), $\frac{1}{n}[c_1 + c_2 + \dots + c_n]$, also converges to c .
- b) Give an example of a sequence that does *not* converge but whose arithmetic mean does converge.
- c) If the arithmetic mean of a sequence converges, must the sequence be bounded? Proof or counterexample.
- d) Let the sequence $b_n > 0$ have the property that $\lim_{n \rightarrow \infty} b_n = b > 0$. Show that the geometric mean also converges: $(b_1 b_2 \dots b_n)^{1/n} \rightarrow b$.
- e) Let $f(t)$ be a continuous function for $0 \leq t < \infty$. If $\lim_{t \rightarrow \infty} f(t) = c$, show that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) dt = c.$$

45. Find the Taylor series centered at $x = 1$ of $\frac{1}{x}$.

46. Let $\{a_n\}$ and $\{b_n\}$ be real sequences. Proof or counterexample.

- If $\sum a_n$ converges, then so does $\sum a_n^2$.
- If $\sum a_n$ and $\sum |b_n|$ both converge, then so does $\sum a_n b_n$.
- If $\sum a_n^2$ converges, then so does $\sum a_n/n$.
- If $\sum a_n$ converges and $a_n \downarrow 0$, then $na_n \rightarrow 0$.

47. Let $\{a_k\}$, $\{b_k\}$ be sequences of real or complex numbers.

- Assuming the a_k are positive, the series $\sum a_k$ converges, and the sequence b_k is bounded, show that the series $\sum a_k b_k$ also converges.
- Give an example showing that the above may be false if the a_k are *not* all positive.

- c) If $S_N := \sum_1^N a_k$ is bounded and the sequence b_k is *monotone decreasing* with $\lim_{k \rightarrow \infty} b_k = 0$, show that the series $\sum a_k b_k$ converges. The standard example is $a_k = (-1)^k$, $b_k = 1/k$. SUGGESTION: Use the following *summation by parts*:

$$\sum_{n+1}^N a_k b_k = [S_N b_N - S_n b_{n+1}] - [S_{n+1}(b_{n+2} - b_{n+1}) + \cdots + S_{N-1}(b_N - b_{N-1})] \quad (1)$$

This analog of integration by parts is due to Abel.

- d) If the series $\sum a_k$ converges and $b_k \rightarrow 0$, does the series $\sum a_k b_k$ necessarily converge? Proof or counterexample.
- e) If the series $\sum a_k$ converges and the sequence b_k is monotone decreasing with $\lim_{k \rightarrow \infty} b_k = b$, does the series $\sum a_k b_k$ necessarily converge? Proof or counterexample.
- f) If $p > 0$, show that the series $\sum_{n=1}^{\infty} n^{-p} \cos n\theta$ converges unless θ is an integer multiple of 2π and $0 < p \leq 1$. [Alternate: Discuss the convergence of the complex series $\sum_{n=1}^{\infty} z^n/n^p$ on the unit circle $|z| = 1$.]
48. [Abel] Assume the real power series $\sum a_n x^n$ converges at $x = 1$. Prove it converges uniformly for $0 \leq x \leq 1$. [SUGGESTION: Use summation by parts (1)]

49. Apply Abel's theorem (Problem 48) to the Taylor series for $\ln x$ and $\arctan x$ to justify the following:

$$a) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2 \qquad b) \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

50. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with integer coefficients a_n . Suppose that for some z with $|z| = 1$ the power series converges. Show that the power series is actually a polynomial.

51. Discuss the convergence of

$$a) \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^2 + n^2} \qquad b) \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(k^2 + n^2)^p}, \quad p > 1.$$

52. For which $p > 0$ does the series $\sum_{k,\ell,n=1}^{\infty} \frac{1}{(k^2 + \ell^2 + n^2)^p}$ converge?

53. Let $f(x) := \sum_1^{\infty} \frac{a_n}{n^x}$. If the sequence a_n is bounded, show that this defines f as a continuous function for all $x > 1$.

54. Consider $f(x) := \sum_{k=1}^{\infty} \frac{(-1)^k \sin kx}{1+k^4}$.

- For which real x is f continuous?
- What can you say about the derivatives of f ?

55. Show that $\sum_{n=1}^{\infty} \frac{\sin(3^n x)}{4^n}$ is differentiable for all real x .

56. If A is a square real matrix, one defines e^A by the power series

$$e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

- If A is a 2×2 diagonal matrix, compute e^A .
- If $A = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$, compute e^A .
- If $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, compute e^A .
- Show that for any A this series converges absolutely (see problem 176). For a space such as \mathbb{R}^n , the simplest norm for matrices A is the *operator norm* $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$, since then $\|AB\| \leq \|A\| \|B\|$ (why?). For a more general norm in a finite dimensional space, one can use that in a finite dimensional space all norms are equivalent (why?).
- If $AB = BA$, show that

$$e^A e^B = e^{A+B}.$$

In particular, this gives $(e^A)^{-1} = e^{-A}$.

57. Let $\zeta(s) = \sum_1^{\infty} \frac{1}{n^s}$ where $s = \sigma + it$ is complex. If $\sigma \geq 2$, Show that $|\zeta(s)| > 0$.

58. For which real numbers c does the series

$$\sum_{n=2}^{\infty} \left(1 - \frac{c}{2}\right) \left(1 - \frac{c}{3}\right) \cdots \left(1 - \frac{c}{n}\right)$$

converge? Here are some hints for one approach:

- a) Show that for any real a_j one has $(1 - a_1)(1 - a_2) \cdots (1 - a_n) \leq e^{-(\sum_1^n a_j)}$.
 b) Show that

$$\int_1^n \frac{1}{x+1} dx < \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^3 \frac{1}{x} dx.$$

- c) Show that

$$\left(1 - \frac{c}{2}\right) \left(1 - \frac{c}{3}\right) \cdots \left(1 - \frac{c}{n}\right) < \left(\frac{2}{n+1}\right)^c.$$

59. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and let a be a positive real number. Show that the function

$$G(x) := \frac{1}{2a} \int_{-a}^a f(x+t) dt$$

is differentiable and has a continuous derivative.

60. Compute the Fourier series for $f(x) = x$, for $-\pi < x < \pi$. Then use the “Pythagorean theorem” (Parseval’s Identity) to compute $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$.

61. Consider the Fourier series (formally, so we don’t yet worry about convergence)

$$f(x) = \sum_{-\infty < k < \infty} c_k e^{ikx} \quad \text{where} \quad c_k \in \mathbb{C} \quad (2)$$

with the complex inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx$.

- a) Show that the functions e^{ikx} for integers k are mutually orthogonal.
 b) Show that $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$. The c_k are the *Fourier coefficients* of f .
 c) If $f \in C^1(|x| \leq \pi)$ and is 2π periodic, show that $|c_k| \leq M/k$ for some constant M independent of k .
 d) Show that formally $\|f\|^2 = \int_{-\pi}^{\pi} |f(t)|^2 = 2\pi \sum_{-\infty < k < \infty} |c_k|^2$ (Pythagoras).
 e) Define the linear map P_N by $(P_N f)(x) := \sum_{|k| \leq N} c_k e^{ikx}$. Show that $P_N^2 = P_N$, that is, $P_N(P_N f) = P_N f$, and also that $\langle P_N f, g \rangle = \langle f, P_N g \rangle$ for any f and g (thus P_N is self-adjoint). These are summarized by saying that the map P_N is an *orthogonal projection*.
 f) Show that $\|f\|^2 = \|P_N f\|^2 + \|(I - P_N)f\|^2$ and hence that $\|P_N f\| \leq \|f\|$.
 g) In any vector space V with an inner product, let \mathcal{W} be a subspace and Pv the orthogonal projection of $v \in V$ into \mathcal{W} . Show that Pv is the point in \mathcal{W} that is closest to v by proving that for any $w \in \mathcal{W}$

$$\|v - w\|^2 = \|v - Pv\|^2 + \|Pv - w\|^2.$$

- h) As an application of part f), let \mathcal{T}_N be the subspace of all functions of the form $\sum_{|k| \leq N} c_k e^{ikx}$. Given a function f , show that for any function $g \in \mathcal{T}_N$ one has $\|f - P_N f\| \leq \|f - g\|$. Thus, the Fourier projection $P_N f$ is closer to f than any other function in \mathcal{T}_N .
- i) Let $D = d/dx$. Show that $P_N D = D P_N$, that is, $P_N(Df) = D(P_N f)$ for all continuously differentiable 2π -periodic functions f .

62. This is a continuation of the previous problem 61 and treats the convergence of Fourier series.

- a) By the Weierstrass approximation theorem, if f is a 2π -periodic function, then given and $\varepsilon > 0$ there is a function g of the form $g = \sum_{|k| \leq N} a_k e^{ikx}$ so that in the uniform norm $\|f - g\|_u < \varepsilon$ (why?).

Combine this with problem 61g) to conclude that $\|f - P_N f\| < 2\pi\varepsilon$ and hence deduce that the Fourier series (2) converges to f in this norm.

- b) Here we consider the uniform convergence of Fourier series. If f is 2π periodic and continuously differentiable, use problems 153a) 61h) to show that

$$\|f - P_N f\|_u \leq \sqrt{2\pi} \|f' - P_N f'\|.$$

Thus, by part a) applied to f' show that for a continuously differentiable function, the Fourier series (2) converges uniformly to f .

63. Let $C^2([-1, 1])$ be the set of functions $u(x)$ that have two continuous derivatives for all $-1 \leq x \leq 1$. Show the following.

- a) The linear space of all $u \in C^2([-1, 1])$ that satisfy $u'' = 0$ is two dimensional.
- b) The linear space of all $u \in C^2([-1, 1])$ that satisfy $u'' + u = 0$ is two dimensional.
- c) The linear space of all $u \in C^2([-1, 1])$ that satisfy $u'' - u = 0$ is two dimensional.
- d) If $a(x) \in C([-1, 1])$, then the linear space of all $u \in C^2([-1, 1])$ that satisfy $u'' + a(x)u = 0$ is *at most* two dimensional.

Assuming the existence of two C^2 functions $\phi(x)$ and $\psi(x)$ that satisfy $u'' + a(x)u = 0$ and the respective initial conditions

$$\phi(0) = 1, \phi'(0) = 0, \quad \text{while} \quad \psi(0) = 0, \psi'(0) = 1,$$

show that the dimension of the space $\{u \in C^2([-1, 1]) \mid u'' + a(x)u = 0\}$ is exactly two.

64. Show that every solution $x(t)$ of the differential equation $x'' + 3x' + 2x = \frac{1}{1+t}$ tends to zero as $t \rightarrow \infty$.

65. Show that the differential equation $x^2y'' - y = 0$ has no Taylor series solution (other than $y(x) \equiv 0$) that converges in some neighborhood of $x = 0$.

66. Let $y(t)$ be a solution near $t = 0$ of the differential equation

$$y' - y^4 = 1, \quad \text{with} \quad y(0) = 0.$$

Show that $y(t)$ *cannot* exist over the whole interval $[0, 2]$.

67. Let $a(t)$ and $f(t)$ be periodic continuous functions with period 2π

a) Show that the equation $u'' = f$ has a 2π -periodic solution if and only if

$$\int_0^{2\pi} f(t) dt = 0.$$

b) Show that the equation $u'' + u = f$ has a 2π -periodic solution if and only if both $\int_0^{2\pi} f(t) \sin t dt = 0$ and $\int_0^{2\pi} f(t) \cos t dt = 0$.

c) Show that the equation $Lu := u'' + au = f$ has a 2π -periodic solution if and only if $\int_0^{2\pi} f(t)z(t) dt = 0$ for all 2π -periodic solutions of $z'' + a(t)z = 0$. [REMARK: These are special cases of the *Fredholm alternative*: the image of L is the orthogonal complement of the kernel of the adjoint operator L^* .]

68. Consider the linear differential operator $Lu := u' + a(t)u$ where $a(t)$ is continuous and periodic with period P , so $a(t+P) = a(t)$ for all real t . Also, let $f(t)$ be a P -periodic continuous function.

a) (*Example*) Show that the equation $u' = f$ has a P periodic solution if and only if $\int_0^P f(t) dt = 0$.

b) Show that the homogeneous equation $Lu = 0$ has a non-trivial P -periodic solution $u(t)$ if and only if $\int_0^P a(t) dt = 0$.

c) If $\int_0^P a(t) dt \neq 0$, show that the inhomogeneous equation $Lu = f$ always has a unique P -periodic solution $u(t)$.

On the other hand, if $\int_0^P a(t) dt = 0$, find a necessary and sufficient condition for $Lu = f$ to have a P -periodic solution. If it has a P periodic solution, is this solution unique?

69. [One goal of this problem is to understand “periodic boundary conditions” for ordinary differential equations. See Part (e) below.]

Say a function $u(t)$ satisfies the differential equation

$$u'' + b(t)u' + c(t)u = 0 \tag{3}$$

on the interval $[0, A]$ and that the coefficients $b(t)$ and $c(t)$ are both bounded, say $|b(t)| \leq M$ and $|c(t)| \leq M$ (if the coefficients are continuous, this is always true for some M).

- Define $E(t) := \frac{1}{2}(u'^2 + u^2)$. Show that for some constant γ (depending on M) we have $E'(t) \leq \gamma E(t)$. [SUGGESTION: use the simple inequality $2xy \leq x^2 + y^2$.]
- Show that $E(t) \leq e^{\gamma t} E(0)$ for all $t \in [0, A]$. [HINT: First use the previous part to show that $(e^{-\gamma t} E(t))' \leq 0$.]
- In particular, if $u(0) = 0$ and $u'(0) = 0$, show that $E(t) = 0$ and hence $u(t) = 0$ for all $t \in [0, A]$. In other words, if $u'' + b(t)u' + c(t)u = 0$ on the interval $[0, A]$ and that the functions $b(t)$ and $c(t)$ are both bounded, and if $u(0) = 0$ and $u'(0) = 0$, then the only possibility is that $u(t) \equiv 0$ for all $t \geq 0$.
- Use this to prove the *uniqueness theorem*: if $v(t)$ and $w(t)$ both satisfy equation

$$u'' + b(t)u' + c(t)u = f(t) \quad (4)$$

and have the same initial conditions, $v(0) = w(0)$ and $v'(0) = w'(0)$, then $v(t) \equiv w(t)$ in the interval $[0, A]$.

- Assume the coefficients $b(t)$, $c(t)$, and $f(t)$ in equation (4) are periodic with period P , that is, $b(t+P) = b(t)$ etc. for all real t . If $\phi(t)$ is a solution of equation (4) that satisfies the *periodic boundary conditions*

$$\phi(P) = \phi(0) \quad \text{and} \quad \phi'(P) = \phi'(0), \quad (5)$$

show that $\phi(t)$ is periodic with period P : $\phi(t+P) = \phi(t)$ for all $t \geq 0$. Thus, the periodic boundary conditions (5) do imply the desired periodicity of the solution.

[NOTE: The ODE $u'' = 1$ has periodic coefficients (with *any* period P) but has no periodic solutions. See the two previous problems.]

70. If the sequence $\{a_n\}$ is bounded and $c > 1$, show that the series $\sum_{n=1}^{\infty} \frac{a_n}{n^x}$ converges absolutely and uniformly in the interval $c \leq x < \infty$.

71. If the series $\sum_0^{\infty} c_n x^n$ converges uniformly for all real x , prove that all but a finite number of the c_n 's must be zero (so the series is a polynomial).

72. Let $\{a_k\}$ be a sequence of complex numbers. If the power series $\sum a_k c^k$ converges for some complex number c , and if $0 < r < |c|$, show that $\sum a_k z^k$ converges absolutely and uniformly in the disk $\{|z| \leq r\}$ in the complex plane.

73. Let $F(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$, and suppose that $\{F_n(x)\}$ is a sequence of continuous functions that converges to F pointwise on the closed interval $[0, 1]$. Prove that the convergence *cannot* be uniform on the *open* interval $(0, 1)$.

74. For which subsets of the real line does the series $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$ converge uniformly?

75. For each of the following give an example of a sequence of continuous functions. Justify your assertions. [A clear sketch may be adequate — as long as it is convincing].

- a) $f_n(x)$ that converge to zero at every x , $0 \leq x \leq 1$, but *not* uniformly.
- b) $g_n(x)$ that converge to zero at every x , $0 \leq x \leq 1$, but $\int_0^1 g_n(x) dx \geq 1$.
- c) $h_n(x)$ converge to zero uniformly for $0 \leq x < \infty$, but $\int_0^{\infty} h_n(x) dx \geq 1$.

76. Let $C(0, 1)$ be the space of continuous functions on the interval $[0, 1]$ and define

$$\|f\|_1 = \int_0^1 |f(x)| dx$$

$$\|f\|_0 = \int_0^1 x|f(x)| dx.$$

Assume that $\|\cdot\|_1$ is a norm on $C(0, 1)$ and show that $\|\cdot\|_0$ is a norm on $C(0, 1)$.

Is $\|\cdot\|_1$ equivalent to $\|\cdot\|_0$?

77. a) Let $\{a_n\}$ be a sequence of real numbers with the property that

$$|a_{k+1} - a_k| \leq \frac{1}{2}|a_k - a_{k-1}|, \quad k = 1, 2, \dots$$

Show that this sequence converges to some real number.

b) NOTATION: $\|\varphi\| = \sup_{0 \leq x \leq 1} |\varphi(x)|$. Using this notation, let $\{u_n(x)\}$ be a sequence of continuous functions for $0 \leq x \leq 1$ with the property that

$$\|u_{k+1} - u_k\| \leq \frac{1}{2}\|u_k - u_{k-1}\|, \quad k = 1, 2, \dots$$

Show that the $\{u_n\}$ converge uniformly to a continuous function.

78. Let $h(x, y)$ and $f(x)$ be continuous for $0 \leq x \leq 2$, $0 \leq y \leq 2$. Let $u_0(x) \equiv 0$ and define $u_k(x)$, $k = 1, 2, \dots$, recursively by the rule

$$u_{k+1}(x) = f(x) + \int_0^c h(x, y)u_k(y) dy.$$

a) Show that if $0 < c \leq 2$ is sufficiently small, then the $u_k(x)$ converge uniformly for $0 \leq x \leq c$ to a continuous function $u(x)$ that satisfies

$$u(x) = f(x) + \int_0^c h(x, y)u(y) dy. \quad (*)$$

b) In the special case where $h(x, y) \equiv 1$ and $f(x) \equiv 1$, solve equation (*) explicitly. [This is easy. Let $\alpha = \int_0^1 u(y) dy$ and then use (*) to solve for α].
From this, show that indeed for some value of c a solution may *not* exist.

79. Let $\varphi(x)$, $x \in \mathbb{R}^n$ be a smooth function with the following properties:

- i). $\varphi(x) > 0$ for $\|x\| < 1$, $\varphi(x) = 0$ for $\|x\| \geq 1$,
- ii). $\int_{\mathbb{R}^n} \varphi(x) dx = 1$.

Let $\varphi_k(x) := k^n \varphi(kx)$. For a continuous function $f(x)$ with $f(x) = 0$ for x outside a compact set \mathcal{K} , define

$$f_k(x) := \int_{\mathbb{R}^n} f(t)\varphi_k(x-t) dt.$$

- a) Show that $\varphi_k(x) = 0$ for $\|x\| \geq 1/k$, and $\int_{\mathbb{R}^n} \varphi_k(x) dx = 1$.
- b) Show that the f_k are smooth functions.
- c) Show that $\lim_{k \rightarrow \infty} f_k(x) = f(x)$, and that this convergence is uniform.
- d) Does the conclusion in part b) remains valid without assuming $\varphi(x) = 0$ for $\|x\| \geq 1$?
- e) Show that the conclusion in part b) remains valid if one assumes only that $f(x)$ is continuous for all x , except that now the convergence is uniform only for x on any compact subset of \mathbb{R}^n .

80. If $f(x)$ is a continuous function that is zero outside of a bounded region, for $t > 0$ let

$$u(x, t) := \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{4t}} dy.$$

Show that $u(x, t)$ converges uniformly to $f(x)$ as $t \searrow 0$.

81. Show that the sequence of functions $f_n(x) := n^3 x^n (1-x)$ does not converge uniformly on $[0, 1]$.

82. Let $h_n(x) = \frac{n^2 x^2}{1 + n^2 x^2}$.

a) On the interval $[-1, 1]$, show $\lim_{n \rightarrow \infty} h_n(x)$ exists but that the convergence is *not* uniform.

b) Is $\lim_{n \rightarrow \infty} \int_{-1}^1 h_n(x) dx = \int_{-1}^1 \lim_{n \rightarrow \infty} h_n(x) dx$? Justify your assertion.

83. Say one has a continuous function $f(x)$ defined for all real x with the property that there is a sequence of polynomials $p_k(x)$ that converge uniformly to f for all x . Thus, given any $\varepsilon > 0$, then for all sufficiently large k we have

$$\sup_{x \in \mathbb{R}} |f(x) - p_k(x)| < \varepsilon.$$

Show that $f(x)$ must itself be a polynomial.

84. [Dini] Let $f_1(x) \geq f_2(x) \geq \dots$ be a decreasing sequence of continuous functions defined for $0 \leq x \leq 1$.

a) If this sequence converges pointwise to zero, show that the convergence is uniform.

b) If this sequence converges pointwise to a continuous function, show that the convergence is uniform.

c) If we replace $0 \leq x \leq 1$ by $0 \leq x < 1$, must the convergence still be uniform? Justify your assertion.

85. Let $f(x)$ be a periodic 2π periodic C^2 function on the real line (so f , f' , and f'' are all 2π periodic). Show that its Fourier series converges uniformly to the function $f(x)$ for all real x .

86. a) If the series $\sum c_n/n^x$ converges (resp. converges absolutely) at $x = x_0$, show that it converges (resp. converges absolutely) for any $x > x_0$.

b) If the series $\sum c_n/n^x$ converges at $x = x_0$, show that it converges absolutely for any $x > x_0 + 1$. Also, give an example showing that if $c < 1$, it may *not* converge absolutely for $x = x_0 + c$.

c) Let $\lambda_n \rightarrow \infty$ be a strictly increasing sequence. If the (Dirichlet) series $\sum_1^N a_n e^{-\lambda_n z}$ converges (resp. converges absolutely) at some complex $z_0 = x_0 + iy_0$, show that it

converges (resp. converges absolutely) for all complex $z = x + iy$ with $x > x_0$. The series $\sum a_n/n^z$ is a special case.

SUGGESTION: Without loss of generality, say $z_0 = 0$. Use summation by parts along with the inequality (which you should prove)

$$|e^{-az} - e^{-bz}| \leq \frac{|z|}{x} (e^{-ax} - e^{-bx}) \quad \text{for real } a < b \text{ and } x > 0.$$

The upshot of your proof should include the observation that the convergence is uniform in any wedge-shaped region for the form $\{z = z_0 + re^{i\theta}\}$ where $|\theta| \leq \text{const} < \pi/2$

87. Assume that $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and let $f_n(x) := \int_0^{x+1/n} g(s) ds$.
- Show that f_n converges to g pointwise.
 - If g is uniformly continuous, show that f_n converges uniformly to g .
 - Give an example of a continuous function g where f_n does *not* converge uniformly to g

88. For $(x, y) \neq (0, 0)$ let

$$f(x, y) = \frac{xy}{x^2 + 2y^2}, \quad g(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}, \quad h(x, y) = \frac{xy^2}{x^2 + y^2}.$$

while $f(0, 0) = 0$, $g(0, 0) = 0$, and $h(0, 0) = 0$.

- Discuss the continuity of these functions.
 - Discuss the differentiability of these functions.
89. Give an example of a smooth function $z = f(x, y)$ defined on all of \mathbb{R}^2
90. a) Find all of the critical points of $(x^2 + 2y^2)e^{-(x^2 + y^2)}$. Here (x, y) is a point in \mathbb{R}^2 . Classify these points as max, min, or saddles.
- b) Let A be an invertible symmetric $n \times n$ matrix with distinct eigenvalues and let $f(x) := \langle x, Ax \rangle e^{-\|x\|^2}$. Here x is a point in \mathbb{R}^n . Find all the critical points of f (there are $2n + 1$ of them).
91. Find the minimum of $Q(x, y, z) := 5x^2 - 2xz + 3y^2 + 5z^2$ for all points (x, y, z) on the unit sphere $x^2 + y^2 + z^2 = 1$.

92. Determine the extremal values of the function given implicitly by $x^4 + y^4 = x^2 + y^2$.
93. Find an example of a smooth function defined on \mathbb{R}^2 with exactly three critical points, all non-degenerate, with one local maximum, one saddle, and one local minimum.
94. Find an example of a smooth function defined on \mathbb{R}^2 with exactly two critical points, both non-degenerate local minima.
95. [J. Pendar] Find all critical points of $p(x,y) := (1+y)^3x^2 + y^2$ and determine if they are local max, min or saddle points or neither. Does f have a global max or min?
96. Find all critical points of $f(x,y) := x^3 - 3x + (x - e^y)^2$ and determine if they are local max, min or saddle points or neither. Does f have a global max or min?
97. Let $\Sigma \subset \mathbb{R}^3$ be a surface (without self-intersections) defined by the points (x,y,z) that satisfy $f(x,y,z) = 0$, where f is a smooth function with $\text{grad } f(x,y,z) \neq 0$ on Σ .
If P is a point *not* on Σ and $Q \in \Sigma$ is a point that is closest to P , show that the line PQ is perpendicular to Σ .
98. Let $f(x,y)$ be a smooth function for all (x,y) in \mathbb{R}^2 . Assume that the origin is *not* on the surface $z = f(x,y)$.
- Show there is at least one point P on this surface that minimizes the distance to the origin.
 - Show that the straight line between the origin and P is perpendicular to the tangent plane to the surface at P .
99. Given any closed set Q in the plane, show there is a continuous function $f \geq 0$ with the property that $f = 0$ only on Q . [In fact, there is a smooth function with this same property, although it is more difficult to construct].
100. Let $h : (x, y, z) \mapsto (x, y, -z)$ be the reflection across the plane $Q := \{z = 0\}$ in \mathbb{R}^3 and let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Assume that the map $f : S^2 \rightarrow S^2$ commutes with h , so $f \circ h = h \circ f$.
- Show that f maps the plane Q into itself.
 - If f is one-to-one and $f(x) = p \in Q$, show that $x \in Q$.
 - Is the above assertion still true if f is not one-to-one? Proof or counterexample.

101. Let $\gamma(t) := (x(t), y(t))$ map $[0, 1] \rightarrow \mathbb{R}^2$ define a smooth curve $\Gamma \subset \mathbb{R}^2$ that does not intersect itself. Assume $\gamma'(t) \neq 0$ for $t \in [0, 1]$. Use γ to find a diffeomorphism $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that maps Γ to the interval $(s, 0) \subset \mathbb{R}^2$, where $s \in [0, 2]$.

102. Let $f: \mathbb{R}^2 \mapsto \mathbb{R}$ be a smooth function with the property that $|\nabla f(x)| = 1$ for all $x = (x_1, x_2)$ in \mathbb{R}^2 . Prove that the integral curves of f , that is, the solutions of $\frac{dx}{dt} = \nabla f(x)$, are straight lines.

103. a) If $u(x_1, x_2, \dots, x_n)$ is a given smooth function, let $u'' := \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)$ be its second derivative (Hessian) matrix. Find all solutions of $\det(u'') = 1$ in the special case where $u = u(r)$ depends only on $r = \sqrt{x_1^2 + \dots + x_n^2}$, the distance to the origin.

b) Let $x = (x_1, x_2, \dots, x_n)$ and A be a square matrix with $\det A = 1$. If $u(x)$ satisfies $\det(u'') = 1$ (see above), and $v(x) := u(Ax)$, show that $\det(v'') = 1$ also. [Remark: the differential operator $\det(u'')$ is interesting because its symmetry group is so large.]

104. a) If $f(x_1, x_2) = h(2x_1 + x_2)$ for some differentiable function $h(t)$, $t \in \mathbb{R}$, show that

$$\frac{\partial f}{\partial x_1} - 2 \frac{\partial f}{\partial x_2} = 0. \tag{6}$$

b) Let $f(x) = f(x_1, \dots, x_n) \in C^2(\mathbb{R}^n)$. Make a linear change of variables

$$x_i = \sum_{j=1}^n s_{ij} y_j,$$

where $S = (s_{ij})$ is an invertible matrix of constants, so $x = Sy$. Write $g(y) := f(Sy) = f(x)$, that is, $g(y)$ is $f(x)$ in the new coordinates. Show that

$$\frac{\partial g}{\partial y_k} = \sum_{j=1}^n \frac{\partial f}{\partial x_j} s_{jk}. \tag{7}$$

If we write $f' = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$ (as a row vector), (7) is just $g'(y) = f'(x)S$ and should remind you of the one variable case $\frac{dg}{dy} = \frac{df}{dx} \frac{dx}{dy}$.

c) [Converse to part (a)] If $f(x_1, x_2)$ is a differentiable function that satisfies (6), find a linear change of variable $y = Ax$ so that $g(y) := f(Sy)$ satisfies $g_{y_2} = 0$.

Use this to conclude that $g(y) = \phi(y_1)$ for some function ϕ depending only on y_1 , and hence that $f(x) = h(2x_1 + x_2)$ for some function h .

If one also knows that $f(0, x_2) = 5 - \sin(3x_2)$, conclude that $f(x) = 5 - \sin 3(2x_1 + x_2)$.

105. [continuation] Let f and g be in part b of the previous problem.

a) Show that

$$\frac{\partial^2 g}{\partial y_k \partial y_\ell} = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} s_{ik} s_{j\ell}. \quad (8)$$

If we use the notation $f''(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)$ (this is a symmetric matrix — why?), show that (8) has the simpler matrix form $g'' = S^T f'' S$, where S^T is the transpose of S .

b) Use the previous part to show that by an appropriate linear change of variable we can always make $g''(0)$ a diagonal matrix.

106. Let (a_{ij}) be a positive definite matrix. If the smooth function $u(x_1, \dots, x_n)$ has a local minimum at a point P , show that $\sum_{1 \leq i, j \leq n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \Big|_{x=P} \geq 0$ [Using Problem 105, first do the special case where P is the origin.]

107. Let (a_{ij}) be a positive definite 2×2 matrix of constants and let L be the linear differential operator

$$Lu := \sum_{1 \leq i, j \leq 2} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

acting on smooth functions $u(x_1, x_2)$. Use Problem 105 to find a linear change of variable $y = Cx$ so that in these new coordinates L has the simpler form

$$Lu = \frac{\partial^2 u}{\partial y_1 \partial y_1} + \frac{\partial^2 u}{\partial y_2 \partial y_2}.$$

Generalize this to where (a_{ij}) is any invertible $n \times n$ constant matrix.

108. Let $w(x)$ and $u(x, y)$ be given smooth functions.

a) If w satisfies $w'' - c(x)w = 0$, where $c(x) > 0$ is a given function, show that w cannot have a local positive maximum. Also show that w cannot have a local negative minimum.

- b) If u satisfies $4u_{xx} + 3u_{yy} - 5u = 0$, show that it cannot have a local positive maximum. Also show that u cannot have a local negative minimum.
- c) Repeat the above for a solution of $4u_{xx} - 2u_{xy} + 3u_{yy} + 7u_x + u_y - 5u = 0$.
- d) If a function $u(x, y)$ satisfies the above equation in a bounded region $\mathcal{D} \in \mathbb{R}^2$ and is zero on the boundary of the region, show that $u(x, y)$ is zero throughout the region.

109. Let $\Omega \subset \mathbb{R}^2$ be a connected open set and $f : \Omega \rightarrow \mathbb{R}$ a differentiable function with the property that $\frac{\partial f(x, y)}{\partial x} = 0$ throughout Ω .

- a) If Ω is convex, prove that $f(x, y) = h(y)$ for some function $h(x)$.
- b) If $\Omega = \{(x, y) : 1 < |x| < 2, y \geq 0\} \cup \{(x, y) : |x| < 2, y < 0\}$, give an example of a function $f \in C^1(\Omega)$ that satisfies $\frac{\partial f(x, y)}{\partial x} = 0$ that does *not* have the form $f(x, y) = h(y)$.

110. For some connected open set $\Omega \in \mathbb{R}^n$, let $f : \Omega \rightarrow \mathbb{R}$ be a real valued function with the property that there is a constant L such that for all $x, y \in \Omega$

$$|f(x) - f(y)| \leq L|x - y|. \quad (9)$$

If this holds we say that the f is *Lipschitz continuous with Lipschitz constant L* .

- a) If f is Lipschitz continuous in Ω with Lipschitz constant L , show that it is uniformly continuous.
- b) If f is differentiable in Ω with $|\nabla f| \leq M$ and Ω is convex, show that f is Lipschitz with Lipschitz constant M .
- c) If in the previous part Ω is not convex, f may not be Lipschitz continuous there. The following example illustrates this. Let Ω be the set from Problem 109b. Define

$$f(x, y) = \begin{cases} y & \text{if } 1 < x < 2, y \geq 1 \\ y^2(2 - y) & \text{if } 1 < x < 2, 0 \leq y \leq 1 \\ 0 & \text{elsewhere in } \Omega. \end{cases}$$

First note that $f \in C^1(\Omega)$. Next, if $1 < a < 2$ and $b > 0$, with $p = (a, b)$, $q = (-a, b)$ show there is no constant L so that $|f(p) - f(q)| \leq L|p - q|$ holds for all b .

111. For some connected $\Omega \in \mathbb{R}$, let $f_k : \Omega \rightarrow \mathbb{R}$ be a sequence of real valued functions that are Lipschitz continuous with the *same* Lipschitz constant L [see Problem 110]. If

the f_k converge pointwise in Ω to some f , show that f is also Lipschitz continuous with the *same* constant L .

If $\Omega \in \mathbb{R}^2$, does the same conclusion hold? Explain.

112. A smooth function $f(x)$ is called *convex* if at any point the curve $y = f(x)$ lies above its tangent line.

- If $f''(x) \geq 0$, show that the curve is convex – and conversely too.
- Show that $e^x \geq 1 + x$ for all (real) x .
- Conversely, if $a > 0$ has the property that $a^x \geq x + 1$ for all real numbers x , show that $a = e$.
- Let $f(x)$ be a smooth convex function for all $x \in \mathbb{R}$. If it is bounded from above: $f(x) \leq \text{const}$, show that f must be a constant.
- Prove the analogue of part a) for a smooth function $f(x)$ of n variables $x = x_1, \dots, x_n$. Here you'll need Taylor series in several variables, and also that the appropriate generalization of $f''(x) \geq 0$ is that the second derivative “Hessian” matrix $\left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right)$ is *positive semi-definite* at every point x .
- Is the analog of part d) true for a function of several variables?

113. Let $f(x)$ be a smooth function of $x = (x_1, \dots, x_n)$ and assume that the Hessian matrix $f''(x)$ at every point is non-singular.

- At a point p , let $\varphi(t) := f(p + tv)$, $t \in \mathbb{R}$, be the restriction of f to the straight line $x = p + tv$. Show that the Hessian matrix $f''(p)$ is positive definite if and only if $\varphi''(0) > 0$ for all directions v . Is it enough if we only check the vectors v in an orthonormal basis?
- Show that f is convex if and only if the restriction of f to every straight line is convex.

114. Say a smooth function $f(x, y)$ defined near the origin on \mathbb{R}^2 has an isolated critical point at the origin and assume that as you approach the origin along any straight line f has a local minimum at the origin.

- If this critical point is nondegenerate (that is, the second derivative matrix is not invertible there), show that the origin is a strict local minimum, so, near the origin $f(x, y) > f(0, 0)$ for all $(x, y) \neq (0, 0)$.
- Give an example of a quartic polynomial showing that if this critical point is degenerate there might be points arbitrarily near the origin where $f(x, y) < f(0, 0)$.

115. The following concerns global issues for critical points of functions of one and several variables. For most people these are not obvious.

- a) There is no smooth function on \mathbb{R}^2 with exactly two critical points, both nondegenerate local minima. Proof or counterexample.
- b) Say the smooth function $f(x)$, defined for all real x has a strict local minimum at the origin and no other critical points. Is $f(0)$ necessarily the global minimum? Why?
- c) Say the smooth function $g(x, y)$, defined for all (x, y) in the plane, has a strict local minimum at the origin and no other critical points. Is $g(0, 0)$ necessarily the global minimum? Why? [*Caution*. For most people this is not obvious].

116. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ be a smooth map.

- a) If $\|\nabla f(x)\| \leq M$ everywhere, show that $\|f(x) - f(y)\| \leq M\|x - y\|$.
- b) Let A be the annular region $A := \{x \in \mathbb{R}^2 : 1 < \|x\| < 2\}$ and $f : A \rightarrow \mathbb{R}^n$ a smooth map. If $\|\nabla f(x)\| \leq M$ for all points in A , estimate $\|f(x) - f(y)\|$ for x and y in A .

117. Let $p(x) = a_n x^n + \dots + a_0$, where $a_n \neq 0$ be a polynomial with real coefficients.

- a) If $x = c$ is a simple root of p , show that it depends smoothly on the coefficients.
- b) Show by examples that at a multiple root this may not be true. In fact, the root may not even be a continuous function.

118. Let $p(x) := (x - 1)(x - 2) \cdots (x - 6) = x^6 - 21x^5 + \dots$ and let $p(x, \varepsilon)$ be the polynomial obtained by replacing $-21x^5$ by $-(21 + \varepsilon)x^5$. Let $x(\varepsilon)$ denote the perturbed value of root $x = 4$, so $x(0) = 4$. Compute the sensitivity of this root as one changes ε , that is, compute $dx(\varepsilon)/d\varepsilon|_{\varepsilon=0}$.

119. The following equations define a map $F : (x, y, z) \mapsto (u, v, w)$:

$$u(x, y, z) = x + xyz^2$$

$$v(x, y, z) = xz^2 + y$$

$$w(x, y, z) = 2x + cz + z^3$$

- a) For which value(s) of the constant c can the system of equations be solved for x , y , z as smooth functions of u , v , w near the point $p = (1, 1, 0)$?

b) Compute the derivative of the inverse at $F(p)$.

120. The following equations define a map $F : (x, y, z) \mapsto (u, v, w)$:

$$u(x, y, z) = x + xyz^2$$

$$v(x, y, z) = y + xy$$

$$w(x, y, z) = z + cx + 3z^2$$

Clearly $F : (1, 1, 0) \mapsto (1, 2, c)$. Write $p = (1, 1, 0)$ and $q = (1, 2, c)$.

a) Compute the derivative $F'(p)$.

b) For which value(s) of the constant c can the system of equations: can be solved for x, y, z as smooth functions of u, v, w near $(1, 1, 0)$? Justify your assertion(s).

c) If c is one of these “good” values, let $G : (u, v, w) \mapsto (x, y, z)$ be the map inverse to F . Compute the derivative $G'(q)$ and use it to compute $\partial y(u, v, w) / \partial v$ at q .

121. Let $y = f(x, u)$ and $z = g(x, u)$ be smooth functions with, say, $f(x_0, u_0) = y_0$ and $g(x_0, u_0) = z_0$.

a) Under what condition(s) can one eliminate u from these equations to express z as $z = F(x, y)$ as a smooth function of x and y near $x = x_0, y = y_0$?

b) Assuming this, then compute $\partial z / \partial x$ and $\partial z / \partial y$ in terms of the derivatives of f and g . To make this computation more specific, assume that

$$f_x(x_0, u_0) = 1, \quad f_u(x_0, u_0) = -2, \quad g_x(x_0, u_0) = -3, \quad \text{and} \quad g_u(x_0, u_0) = 4.$$

122. If a square matrix A of real numbers is sufficiently close to the identity matrix, show that it has a real square root B , that is, $B^2 = A$.

123. Let $f(x)$ be smooth near the origin. If $f(0) > 0$, show that near the origin there is a smooth function g such that $f(x) = g^2(x)$.

124. Let $f(x)$ be smooth for $x \in \mathbb{R}$ near the origin.

a) If $f(0) = 0$, show that near the origin there is a smooth function $g(x)$ so that $f(x) = xg(x)$. [This is of course obvious if f has a convergent Taylor series.]

b) If $f'(0) = 0$ but $f''(0) > 0$, show that near the origin there is a smooth function $g(x)$ so that $f(x) - f(0) = x^2g(x)$ and a smooth function h so that $f(x) = h^2(x)$.

125. Generalize the previous problems to a smooth function $f(x)$ where $x = (x_1, x_2) \in \mathbb{R}^2$.

- a) Here the assumption is that $f(0) = 0$.
 b) Here assume that $\nabla f(0) = 0$ but that the 2×2 second derivative (or Hessian) matrix,

$$f'' := \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right),$$

is non-singular. The conclusion should imply that there are new (smooth) coordinates $u = u(x_1, x_2)$ and $v = v(x_1, x_2)$ so that near the origin $f(x) - f(0) = \pm u^2 \pm v^2$, where the \pm signs depend on the “signature” of $\text{Hess}(f)$ at the origin. [The *signature* of a symmetric matrix is the number of positive and negative eigenvalues].

- c) Generalize this problem to functions $f(x_1, \dots, x_n)$.

126. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth map with the property that $F(0) = 0$ and $f'(0) = I$, where I is the identity map. Show there is an $n \times n$ matrix $A(x)$ whose entries depend smoothly on x so that $f(x) = A(x)x$ for all x near the origin. Also show that $A(0) = I$.

127. [The Morse Lemma for a smooth function $z = f(x, y)$ of two real variables.] Assume

$$f(0, 0) = 0, \quad f'(0, 0) = 0, \quad f''(0, 0) \text{ is positive definite}$$

and show that near the origin there are new coordinates $u = u(x, y)$, $v = v(x, y)$ so that in these coordinates $z = u^2 + v^2$. [SUGGESTION: First do the special case where f already has the special form

$$f(x, y) = x^2 h_{11}(x, y) + 2xy h_{12}(x, y) + y^2 h_{22}(x, y),$$

for smooth functions $h_{ij}(x, y)$ where $h_{11}(0, 0) = 1$, $h_{12}(0, 0) = 0$, and $h_{22}(0, 0) = 1$].

128. a) Let $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth map with the property that its first partial derivatives are uniformly bounded for all points in \mathbb{R}^n . Show there is a constant k such that for all x and y in \mathbb{R}^n

$$\|G(x) - G(y)\| \leq k\|x - y\|. \quad (10)$$

b) Let $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth map with the property (10) for all points in \mathbb{R}^n . Let $F(x) := x + cG(x)$. If the real constant $c > 0$ is chosen sufficiently small, show that F is a diffeomorphism of \mathbb{R}^n .

129. Let $0 < a \leq b$ be real numbers. Define

$$M = M(a, b) = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} dt.$$

Show that $a \leq M \leq b$. [REMARK: the length of the ellipse $x = a \sin t$, $y = b \cos t$ is $2\pi M$.]

130. Show that the improper integral $\int_0^\infty \frac{\sin x}{\sqrt{x}} dx$ is convergent, but not absolutely convergent.

131. For which real numbers s do the following improper integrals converge? Why?

$$I(s) = \int_0^\infty t^s |\sin t| dt \quad \text{and} \quad J(s) = \int_0^\infty t^s \sin t dt$$

132. Let $f: [0, 1] \rightarrow \mathbb{R}$ be a continuous function.

a) Show that $\lim_{\lambda \rightarrow \infty} \int_0^1 f(x) \sin(\lambda x) dx = 0$.

b) Compute $\lim_{\lambda \rightarrow \infty} \int_0^1 |\sin(\lambda x)| dx$.

c) (generalization) If $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is continuous with period P , show that

$$\lim_{\lambda \rightarrow \infty} \int_0^1 f(x) \varphi(\lambda x) dx = \bar{\varphi} \int_0^1 f(x) dx,$$

where $\bar{\varphi} := \frac{1}{P} \int_0^P \varphi(t) dt$ is the average of φ over one period.

133. Let $f(t)$ be continuous for $-\infty < t < \infty$, assume that $\int_{-\infty}^\infty |f(t)| dt < \infty$, and define $F(x) = \int_{-\infty}^\infty e^{ixt} f(t) dt$.

Show that $F(x)$ is continuous for all real x .

134. a) Describe the set of continuous functions $\{f(x)\}$ for $x \in [-\pi, \pi]$:

$$\int_{-\pi}^{\pi} f(x) \sin kx dx = 0, \quad k = 1, 2, \dots$$

b) Describe the set of continuous functions $\{f(x)\}$ for $x \in [0, \pi]$:

$$\int_0^{\pi} f(x) \sin kx dx = 0, \quad k = 1, 2, \dots$$

135. a) Let $f(x) \in C([0, 1])$. If $\int_0^1 f(x) x^n dx = 0$ for all $n = 0, 1, 2, \dots$, show that f must be identically zero.

- b) Let $f(x) \in C([0, 1])$. If $\int_0^1 f(x)x^{2k} dx = 0$ for all $k = 0, 1, 2, \dots$, what can you conclude?
- c) Let $f(x) \in C([-1, 1])$. If $\int_{-1}^1 f(x)x^n dx = 0$ for all $n = 0, 1, 2, \dots$, what can you conclude?
- d) Let $f(x) \in C([-1, 1])$. If $\int_{-1}^1 f(x)x^{2k} dx = 0$ for all $k = 0, 1, 2, \dots$, what can you conclude?

136. Let $f(x)$ be a continuous function for $0 \leq x \leq 10$. Compute

$$\lim_{n \rightarrow \infty} \int_0^n f\left(\frac{t}{n}\right) e^{-t} dt.$$

137. Find a function $f \in C(\mathbb{R})$ so that $f(x) \geq 0$, and $\int_0^\infty f(x) dx < 2$, but $f(x)$ is *not* bounded. [REMARK: This is in contrast to the infinite series result that if $\sum a_k$ converges, then $a_k \rightarrow 0$.]

138. Compute $\lim_{n \rightarrow \infty} n^k \int_0^1 \frac{x^{n-1}}{x+1} dx$ for $k = 0$ and $k = 1$.

139. Let $f(x)$ be a continuous function for $0 \leq x \leq 1$. Evaluate $\lim_{n \rightarrow \infty} n \int_0^1 f(x)x^n dx$. (Justify your assertions.)

140. Let $f(x)$ be a continuous function for $0 \leq x \leq 10$. Find all real numbers c for which $Q_c(f) := \lim_{n \rightarrow \infty} n^c \int_0^{10} f(x)e^{-nx} dx$ exists. If the limit $Q_c(f)$ exists, compute it.

141. Let $f(x)$ be a real-valued continuous function defined for all $x \geq 0$. If $\lim_{x \rightarrow \infty} f(x) = c$, show that for any $a > 0$ the sequence

$$Q_n := \int_0^a f(x^n) dx, \quad n = 1, 2, 3, \dots$$

converges.

142. Let $f(x)$ be a uniformly continuous function in $[0, \infty)$ and assume that $\int_0^\infty f(x) dx$ exists. Is it true that $\lim_{x \rightarrow \infty} f(x) = 0$? Justify your response.

143. Let $f_n(x) = n^2 x^{n-1} (1-x)$. Prove that $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx \neq \int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx$.

144. Let $p(x)$ be a real polynomial of degree n . The following uses the inner product $\langle f, g \rangle := \int_0^1 f(x)g(x) dx$.

- If p is orthogonal to the constants, show that p has at least one real zero in the interval $\{0 < x < 1\}$.
- If p is orthogonal to all polynomials of degree at most one, show that p has at least two distinct real zeros in the interval $\{0 < x < 1\}$.
- If p is orthogonal to all polynomials of degree at most $n-1$, show that p has exactly n distinct real zeros in the interval $\{0 < x < 1\}$.

145. In number theory, the function $\text{Li}(x) := \int_2^x \frac{dt}{\log t}$ arises in estimating the number of primes less than x . Show that $\text{Li}(x)$ is asymptotically equal to $x/\log x$ for large x , that is

$$\lim_{x \rightarrow \infty} \frac{\text{Li}(x)}{\frac{x}{\log x}} = 1,$$

146. If $f(t)$ is a bounded piecewise continuous function, so $|f(t)| \leq c$ for $t \geq 0$, define its Laplace transform $F(x)$ for $x > 0$, as

$$F(x) := \int_0^{\infty} e^{-xt} f(t) dt.$$

Show that for $x > 0$ the function $F(x)$ is differentiable and the derivative is continuous.

147. a) Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere in \mathbb{R}^3 . Compute $\int_{S^2} x^2 d\sigma$, where $d\sigma$ is the element of area on S^2 .

b) Let $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_{n+1}^2 = 1\}$ be the unit sphere in \mathbb{R}^{n+1} . Compute $\int_{S^n} x_1^2 d\sigma$, where $d\sigma$ is the element of “area” on S^n . [Your answer will involve $\sigma_n := \text{Area of } S^n$.

148. Let x be positive and consider $\int_0^{\infty} e^{-xy} dy$. State a theorem showing that differentiation under the integral sign is valid for $x \geq c$, where $c > 0$. Differentiate n times to deduce

$$\int_0^{\infty} y^n e^{-xy} dy = \frac{n!}{x^{n+1}}, \quad \text{for } x \geq c.$$

[Note that if $x = 1$ we get $\Gamma(n + 1) = n!$.]

149. Repeat the previous problem for $\int_0^\infty \frac{dy}{x^2 + y^2}$ for $x \geq c > 0$.

a) Differentiate n -times to find

$$\int_0^\infty \frac{dy}{x^2 + y^2} = \frac{\pi}{2} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2 \cdot 4 \cdot 6 \cdots (2n - 2)} \right) \frac{1}{x^{2n-1}}.$$

b) Let $x = \sqrt{n}$ and *prove* that as $n \rightarrow \infty$ the integral on the left tends to $\int_0^\infty e^{-y^2} dy$, which is known to be $\sqrt{\pi}/2$.

c) Use the above to prove:

$$\lim_{n \rightarrow \infty} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2 \cdot 4 \cdot 6 \cdots (2n - 2)} \right) \sqrt{n} = \frac{1}{\sqrt{\pi}}.$$

150. a) Show that the integral $F(x) := \int_0^\infty e^{-xt} \frac{\sin t}{t} dt$ converges uniformly for $x \geq 0$ and that $G(x) := \int_0^\infty e^{-xt} \sin t dt$ converges uniformly if $x \geq \varepsilon > 0$.

b) State a theorem that allows you to conclude $F'(x) = -G(x)$.

c) Integrating G to deduce that $F(x) = \text{arc cot } x$. Then let $x \downarrow 0$ to conclude that

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

151. Compute $F(\alpha) := \int_0^\infty e^{-\alpha x^2} \cos x dx$, assuming $\alpha > 0$. [Hint: Find F' — and justify.]

152. Let $\gamma: \mathbb{R} \rightarrow \mathbb{R}^3$ define a smooth curve. Show that $\left\| \int_0^1 \gamma(t) dt \right\| \leq \int_0^1 \|\gamma(t)\| dt$.

153. Let $f(x)$, $a \leq x \leq b$ be a smooth function.

a) If $f(c) = 0$ for some $a \leq c \leq b$, show that

$$|f(x)| \leq \int_a^b |f'(t)| dt \leq \sqrt{b-a} \left[\int_a^b |f'(t)|^2 dt \right]^{1/2}.$$

and hence, using the uniform norm $\|f\|_{\text{unif}} := \max_{a \leq x \leq b} |f(x)|$,

$$\|f\|_{\text{unif}} \leq \int_a^b |f'(t)| dt \leq \sqrt{b-a} \left[\int_a^b |f'(t)|^2 dt \right]^{1/2}.$$

- b) If $\int_a^b f(t) dt = 0$ (this replaces the assumption $f(c) = 0$), show that the above inequality still holds.
- c) Use the result of part b) to show that there are constants α and β so that for *any* smooth f

$$\|f\|_{\text{unif}} \leq \alpha \int_a^b [|f'(t)| + |f(t)|] dt \leq \beta \left[\int_a^b [|f'(t)|^2 + |f(t)|^2] dt \right]^{1/2}.$$

154. Let $f(x) \in C([a, b])$. Show that

$$\exp \left[\frac{1}{b-a} \int_a^b f(x) dx \right] \leq \frac{1}{b-a} \int_a^b \exp[f(x)] dx$$

Similarly, let $\mathcal{D} \in \mathbb{R}^2$ be a bounded region whose boundary is smooth enough that the integral below all exist. Define the *average*, \bar{h} , of a continuous function h by $\bar{h} = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} h dA$, where dA is the usual element of area. Show that

$$\frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} e^h dA \geq e^{\bar{h}}$$

155. Let α be an irrational real number and let $f(\theta)$ be a continuous 2π periodic function, $0 \leq \theta \leq 2\pi$. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(2\pi k\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta.$$

[REMARK: This is a stronger version of problem 43 since it shows that the points $2\pi k\alpha$, $k = 1, 2, \dots$ are not only dense but are *equidistributed* around the circle. Here equidistributed means that in any two intervals, the limit of the ratios of the number of points in these intervals is proportional to the length of the interval.]

156. The *Gauss curvature* of a surface $z = u(x, y)$ is

$$K(x, y) = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + |\nabla u|^2)^2}.$$

If $K(x, y) \geq c > 0$ for (x, y) in a connected region Ω , show that $c \text{Area}(\Omega) \leq \pi$. Thus for instance, if $K \equiv 1$ then $\text{Area}(\Omega) \leq \pi$, as happens for a hemisphere of radius 1. [SUGGESTION: Integrate K over Ω and make the change of variable $(p_1, p_2) = (u_x, u_y) = \nabla u$.]

157. a) Let A be a symmetric matrix with eigenvalue λ and corresponding eigenvector v , $Av = \lambda v$. Show that

$$\lambda = \frac{\langle v, Av \rangle}{\|v\|^2}.$$

- b) If A is positive definite with positive definite square root P , so $A = P^2$, show that

$$\lambda = \frac{\|Pv\|^2}{\|v\|^2}.$$

- c) A standard ingredient in many problems involves the eigenvalues λ and corresponding eigenfunctions u of the Laplacian $\Delta = \nabla^2$,

$$-\Delta u = \lambda u \quad \text{in } \mathcal{D} \quad \text{with} \quad u = 0 \quad \text{on } \mathcal{B},$$

Here \mathcal{D} in \mathbb{R}^2 is a bounded region with boundary \mathcal{B} . As usual, to be useful one wants numbers λ so that there is a solution u other than the trivial solution $u \equiv 0$. Show that

$$\lambda = \frac{\iint_{\mathcal{D}} |\nabla u|^2 dA}{\iint_{\mathcal{D}} u^2 dA}.$$

[SUGGESTION: Use the divergence theorem].

In particular, deduce that $\lambda > 0$.

158. a) For $k \geq 0$, let $J_k = \int_0^\infty t^k e^{-t^2} dt$. Show that $J_k = \frac{k-1}{2} J_{k-2}$.

Using the values of $J_0 = \sqrt{\pi}/2$ and $J_1 = ?$, find a formula for J_k , $k = 2, 3, \dots$

- b) Let $d\sigma_{n-1}$ denote the element of “area” on the unit sphere S^{n-1} (the points $x \in \mathbb{R}^n$ where $\|x\| = 1$) and Ω_{n-1} the area of the whole sphere (so $\Omega_2 = 4\pi$).

Using spherical coordinates in \mathbb{R}^n , justify the following computation:

$$\iint_{\mathbb{R}^n} e^{-\|x\|^2} dx = \iint_{S^{n-1}} \left(\int_0^\infty e^{-r^2} r^{n-1} dr \right) d\sigma_{n-1} = \Omega_{n-1} \int_0^\infty e^{-r^2} r^{n-1} dr$$

- c) Since we also know that $\iint_{\mathbb{R}^n} e^{-\|x\|^2} dx = \pi^{n/2}$, use the computation of the integral J_k in part a), give an explicit formula for Ω_n , $n = 2, 3, 4, 5$.

159. a) Compute

$$\iint_{\mathbb{R}^2} \frac{dx dy}{(1 + 4x^2 + 9y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx dy}{(1 + x^2 + 2xy + 5y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx dy}{(1 + 5x^2 - 4xy + 5y^2)^2}.$$

b) Compute $\iint_{\mathbb{R}^2} \frac{dx_1 dx_2}{[1 + \langle x, Cx \rangle]^2}$, where C is a positive definite (symmetric) 2×2 matrix, and $x = (x_1, x_2) \in \mathbb{R}^2$.

c) Let $h(t)$ be a given function and say you know that $\int_0^\infty h(t) dt = \alpha$. If C be a positive definite 2×2 matrix. Show that

$$\iint_{\mathbb{R}^2} h(\langle x, Cx \rangle) dA = \frac{\pi\alpha}{\sqrt{\det C}}.$$

d) Compute $\iint_{\mathbb{R}^2} e^{-(5x^2 - 4xy + 5y^2)} dx dy$.

e) Compute $\iint_{\mathbb{R}^2} e^{-(5x^2 - 4xy + 5y^2 - 2x + 3)} dx dy$.

f) Generalize part c) to obtain a formula for

$$\iint_{\mathbb{R}^n} h(\langle x, Cx \rangle) dV,$$

where now C be a positive definite $n \times n$ matrix. The answer will involve some integral involving h and also the “area” of the unit sphere S^{n-1} in \mathbb{R}^n .

160. Let A be an $n \times n$ positive definite matrix, $b \in \mathbb{R}^n$ a given vector, and $c \in \mathbb{R}$ a scalar. Define the quadratic polynomial $Q(x)$ as

$$Q(x) = \langle x, Ax \rangle + 2\langle b, x \rangle + c.$$

a) [COMPLETING THE SQUARE]. Show that by an appropriate choice of the vector w , after the change of variable $x = y - w$ (a translation) you can eliminate the linear term in $Q(x)$ to obtain

$$Q = \langle y, Ay \rangle + c - \langle b, A^{-1}b \rangle.$$

[SUGGESTION: First do the case $n = 1$.]

b) Use this to generalize the previous problem to obtain the formula

$$\iint_{\mathbb{R}^n} e^{-[\langle x, Ax \rangle + \langle b, x \rangle + c]} dx = \frac{\pi^{n/2}}{\sqrt{\det A}} e^{\langle b, A^{-1}b \rangle - c}.$$

161. Let S be any symmetric matrix and A a positive definite matrix.

a) Show that

$$\iint_{\mathbb{R}^n} \langle x, Sx \rangle e^{-\|x\|^2} dx = \frac{1}{2} \pi^{n/2} \text{trace}(S).$$

b) Show that

$$\iint_{\mathbb{R}^n} \langle x, Sx \rangle e^{-\langle x, Ax \rangle} dx = \frac{\pi^{n/2} \text{trace}(SA^{-1})}{2\sqrt{\det A}}.$$

162. Let f be a Lebesgue integrable function on $[0, 1]$ and assume that $0 \leq \int_A f d\mu \leq m(A)$ for all measurable $A \subset [0, 1]$. Show that $0 \leq f \leq 1$ almost everywhere.

163. Let f be a Riemann integrable function and define $F(x) := \int_a^x f(t) dt$. Show that F is differentiable almost everywhere and $F' = f$ almost everywhere. (This is also true for Lebesgue integrable functions but much harder to prove).

164. a) Make sense of the following: “Let $\mathcal{D}_t \subset \mathbb{R}^2$ be a family of bounded regions in the plane with smooth boundaries. These regions depend smoothly on a real parameter t .”

b) As a test of the effectiveness of your definition, use it to compute the derivative of

$$J(t) := \iint_{\mathcal{D}_t} f(x, y) dx dy,$$

at $t = 0$ where $f(x, y)$ is a given smooth function and \mathcal{D}_t is the family of ellipsoids $x^2 + (1 + 3t)y^2 = 1$.

165. a) Let $f : I \rightarrow \mathbb{R}$ be a non-decreasing function, where $I = [a, b]$ is a bounded interval. Prove that the set of points where f is discontinuous is either finite or countable.

b) In this assertion still valid if I is replaced by a non-compact interval? [Proof or counterexample.]

166. Let C denote the set of all continuous real-valued functions on the interval $[0, 1]$. Show that C is a vector space and that it is infinite dimensional.

167. Let $B := \{f \in C([0, 1]) : f(x) > 0 \text{ for all } x \in [0, 1]\}$. Show that B is an open subset of $C([0, 1])$. What is the closure of B ?

168. Let X and Y be metric spaces with the metrics d_X and d_Y , respectively, and let $W = X \times Y$ be the Cartesian product space of all pairs of points (x, y) , where $x \in X$

and $y \in Y$. The simplest example is where $X = Y = \mathbb{R}$ so $W = \mathbb{R}^2$. If $p = (x_1, y_1)$ and $q = (x_2, y_2)$, define the following metric on W :

$$d(p, q) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

- a) Show that $d(p, q)$ has all the properties of a metric, so W is a metric space.
- b) If both X and Y are bounded, show that W is bounded.
- c) If both X and Y are compact, show that W is compact.
- d) If both X and Y are complete, show that W is complete.
- e) If both X and Y are connected, show that W is connected.
- f) Say we define a different metric on W :

$$d_2(p, q) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

Show that this is also a metric and that this metric is *equivalent* to the previous one.

169. Let Q be the interior of the unit square $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ from which the following vertical line segments have been deleted:

$$\{x = 1/2^{2j-1}, 0 \leq y \leq 1 - 1/2^{2j-1}\}, \quad \{x = 1/2^{2j}, 1/2^{2j} \leq y \leq 1\}, \quad j = 1, 2, \dots$$

(draw a sketch). Clearly Q is bounded and simply connected. Show that it has the property that given any $L \in \mathbb{R}$ there are points in Q such that any curve in Q between them has length greater than L . Thus the diameter of L is infinite. [The points on the line segment $x = 0, 0 \leq y \leq 1$ are called *inaccessible boundary points*.]

170. Show that every connected metric space with at least two points is uncountable.
171. Show that in a connected metric space (X, d) the only non-empty set which is both open and closed set is all of X .
172. a) Let $(V, |\cdot|)$ be a vector space with a norm. Show that the norm comes from an inner product $\langle \cdot, \cdot \rangle$, (that is, $|x|^2 = \langle x, x \rangle$) if and only if the norm satisfies the *parallelogram law*:

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$

- b) Show that the p -norm on $C^0([0, 1])$ defined by $\|f\|_p = [\int_0^1 |f|^p]^{1/p}$, $p \geq 1$ comes from an inner product if and only if $p = 2$. [For simplicity, assume the vector space is over the reals].
- c) Show that $\|\cdot\|$ as defined in (b) is not a norm when $p < 1$.

173. Let T be a complete metric space with metric ρ and S any set. Denote by \mathcal{B} the set of all functions from S to T with the (possibly infinite) metric

$$\sigma(f, g) = \sup_{x \in S} \rho(f(x), g(x)).$$

Show that \mathcal{B} with this metric is complete.

174. (continuation) Let S be a metric space with metric d and T be a complete metric space with metric ρ . Denote by \mathcal{C} the space of all continuous functions from S to T with the same (possible infinite) metric σ as in the previous problem. Show that \mathcal{C} with this metric is complete.

175. In a metric space M let $d(x, y)$ denote the distance. A sequence x_j is called a *fast Cauchy sequence* if $\sum_j d(x_{j+1}, x_j) < \infty$.

- a) In \mathbb{R} give an example of a fast Cauchy sequence and also of a Cauchy sequence that is *not* fast.
- b) Show that every fast Cauchy sequence is indeed a Cauchy sequence.
- c) If there is a constant $0 < c < 1$ such that for all i

$$d(x_{i+1}, x_i) < cd(x_i, x_{i-1})$$

show that x_j is a fast Cauchy sequence.

- d) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 function. Given any real x_0 define x_j recursively by the rule $x_{j+1} = f(x_j)$. If the first derivative of f is uniformly bounded, so for all x we have $|f'(x)| < c$ for some constant $0 < c < 1$, show that x_j is a fast Cauchy sequence.
- e) Show that every Cauchy sequence has a fast Cauchy subsequence.

176. Let $\{X_j\}$ be a sequence of vectors in a normed linear space V . We say the infinite series $\sum X_j$ *converges* if the sequence of vectors $S_N = \sum_{j=1}^N X_j$ converges (to some point of V), while we say it *converges absolutely* if the series of real numbers $\sum \|X_j\|$ converges.

- a) If V is complete and $\sum X_j$ converges absolutely, show that it converges. Also, give an example showing that the assumption “ V is complete” is needed.
- b) Give an example showing that a series may converge – but not absolutely.
- c) If *every* absolutely convergent series in V converges, show that V is complete.

177. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $1 \leq p < \infty$, define $\|x\|_p := [\sum_{j=1}^n |x_j|^p]^{1/p}$ and $\|x\|_\infty := \max_{1 \leq j \leq n} |x_j|$.

- a) If $n = 2$, sketch the unit balls $\|x\|_p = 1$ for $p = 1, 2, 4$ and ∞ .
- b) Show that $\lim_{p \rightarrow \infty} \|x\|_p = \|x\|_\infty$.

178. For any two sets $S, T \subset \mathbb{R}^n$, define

$$\text{dist}(S, T) = \inf\{\text{dist}(x, y) \mid x \in S, y \in T\}$$

- a) If S is compact, T is closed and $S \cap T = \emptyset$, prove that $\text{dist}(S, T) > 0$.
- b) Does (a) remain true if S and T are any two disjoint closed subsets of \mathbb{R}^n ?

179. Prove that a set $K \subset \mathbb{R}^n$ is compact if and only if for each continuous function $f : K \rightarrow \mathbb{R}$, the set $f(K)$ is bounded.

180. Let (X, d) be a complete metric space and $A \subset X$ a dense subset. If $f : A \rightarrow \mathbb{R}$ is a uniformly continuous function, show that f has a unique continuous extension to all of X .

181. Let $S \subset \mathbb{R}^n$ be an open set. Show that S can be covered by a *countable* number of open balls.

182. Let $C \subset \mathbb{R}^n$ be *any* closed set.

- a) Show there is a continuous function f that is zero on C and $f(x) > 0$ for all $x \notin C$.
- b) Show there is a smooth (C^∞) function f that is zero on C and $f(x) > 0$ for all $x \notin C$. [SUGGESTION: Take a countable number of open balls B_j that cover $\mathbb{R}^n - C$ and let $\phi_j(x) > 0$ for $x \in B_j$, with $\phi_j(x) = 0$ outside of B_j . Then define $f(x) = \sum_j (c_j \phi_j(x))$, where $c_j > 0$ is a decreasing sequence of numbers chosen so that $f(x)$ is smooth.]

183. [E. Borel] Let a_0, a_1, \dots be *any* sequence of real numbers. Show there is a smooth function $f(x)$ with the property that a_n is its n^{th} Taylor coefficient: $a_n = \frac{1}{n!} f^{(n)}(x)|_{x=0}$.

[Last revised: November 28, 2015]