Math 361 — **Exam 3** — Jerry Kazdan

December 8, 2015 — 10:30 – 12:00

DIRECTIONS This exam has two parts. Part A has 6 shorter questions (7 points each so 42 points) while Part B had 5 problems (15 points each, so 75 points for this part). Maximum total score is thus 117 points.
Closed book, no calculators etc. – but you may use one $3'' \times 5''$ card with notes on both sides.

Remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 10:30 and ends at 12:00.
Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: There are 6 short answer questions, 7 points each so 42 points for this part.

A–1. Give an example of a sequence of smooth functions $f_k : \mathbb{R} \to \mathbb{R}$ so that the infinite series $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly for all real x but the series $\sum_{k=1}^{\infty} f_k'(x)$ *diverges* at $x = 0$.

SOLUTION: One example is $\displaystyle\sum_1^{\infty} \frac{\sin k^3 x}{k^2}$. Since $|\sin k^3 x| \le 1$, this series converges uniformly by the Weierstrass M test.

The derivative series is $\displaystyle\sum_1^{\infty} k \cos k^3 x$ which clearly diverges at $x = 0$.

A–2. If the radius of convergence of the real power series $\sum_{n=0}^{\infty} a_n x^n$ is $R < \infty$, then $a_n R^n \to 0$. Proof or counterexample.

SOLUTION: Counterexample: The geometric series $\sum_0^{\infty} 3x^n$ so here $a_n = 3$ and $R = 1$. Another, more troublesome, example is $\sum n x^n$ whose radius of convergence $R = 1$.

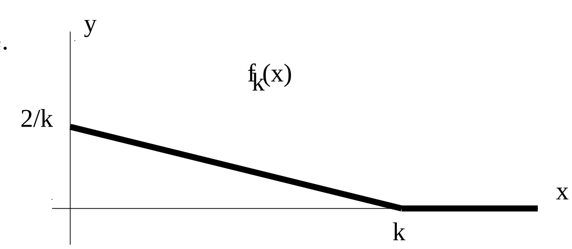
A–3. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a smooth curve in $\mathbb{R}^3$ parameterized by arc length s, so $\|\gamma'(s)\| = 1$. Show that the vector $\gamma''(s)$ is perpendicular to the tangent vector, $\gamma'(s)$.

SOLUTION: Since $\langle \gamma'(s), \gamma' s) \rangle = \|\gamma'(s)\|^2 = 1$, taking the derivative we find $2\langle \gamma'(s), \gamma''(s) \rangle = 0$. Therefore $\gamma''(s) \perp \gamma'(s)$.

A–4. Let $f_k : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions. Assume that $f_k(x) = 0$ for all $x \ge k$ so the integrals $\int_0^{\infty} f_k(x)\,dx$ all exist. If f_k converges uniformly to 0 on the set $x \ge 0$, does $\int_0^{\infty} f_k(x)\,dx$ converge to 0? Proof or counterexample.

SOLUTION: This is the graph of one counterexample. Note that

$$\int_0^{\infty} f_k(x)\,dx = 1.$$



A–5. The *Gamma function* is defined by the improper integral $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx$. For which real values of t does this improper integral exist? Why?

SOLUTION: This has possible difficulties: at $x = 0$ and at $x = \infty$. Write

$$\Gamma(t) = \int_0^1 x^{t-1}e^{-x}\,dx + \int_1^\infty x^{t-1}e^{-x}\,dx$$

In the first integral, we need to careful at $x = 0$ where the integrand might blow-up. The second integral requires caution because the region of integration is unbounded.

For the first integral, since $\int_0^1 x^c\,dx$ converges for any $c > -1$, this integral converges for any $t > 0$.

The second integral converges near $x = \infty$ because the exponential decay (e^{-x}) dominates any possible growth of x^{t-1}. In greater detail, $x^{t-1}e^{-x} = [x^{t-1}e^{-x/2}]e^{-x/2}$. For $x \ge 1$ the first factor tends to zero and hence is bounded while the second decays quickly. Consequently, this integral converges for all real t.

Bottom line: this improper integral converges for all $t > 0$.

A–6. Must the boundary of a set of measure zero have measure zero? Proof or counterexample.

SOLUTION: Counterexample: Let S be the set of all rational numbers in the interval $0 \le x \le 1$. Then the boundary of S is the whole interval $0 \le x \le 1$, which certainly does not have measure zero.

PART B 5 questions, 15 points each (so 75 points total).

B–1. Let $P_1, P_2, \ldots, P_k$ be distinct points in $\mathbb{R}^n$.

a) Find the point $X_0 \in \mathbb{R}^n$ that minimizes the function

$$Q(X) = \|X - P_1\|^2 + \|X - P_2\|^2 + \cdots + \|X - P_k\|^2.$$

SOLUTION: Since $\nabla\|X - P\|^2 = 2(X - P)$, then

$$\nabla Q(X) = 2[(X - P_1) + \cdots + (X - P_k)] = 2[kX - (P_1 + P_2 + \cdots + P_k)].$$

Therefore the only critical point of Q is $X_0 = (P_1 + P_2 + \cdots + P_k)/k$.

b) Why is the point X_0 you just found the *global* minimum of $Q(X)$? That is, why is $Q(X_0) \le Q(X)$ for all $X \in \mathbb{R}^n$? [There are several completely different ways to show this.]

SOLUTION: Method 1). Since from part a) the Hessian matrix $Q''(X) = 2kI$, it is positive deffinite so the graph of $Q(X)$ is convex. Therefore at any point it lies above its tangent plane. In particular, it lies above its tangent plane at X_0. Because $\nabla Q(X_0) = 0$ this tangent plane is horizontal.

More briefly, by Taylor's theorem with two terms

$$Q(X) = Q(X_0) + \nabla Q(X_0)(X - X_0) + \cdots \geq Q(X_0).$$

Method 2). Since $Q(X)$ is a quadratic polynomial, we complete the square:

$$\begin{aligned} Q(X) =& \sum_1^k \left(\|X\|^2 - 2\langle X,\, P_j\rangle + \|P_j\|^2 \right) \\ =& k\|X\|^2 - 2\langle X,\, \sum P_j\rangle + \sum \|P_j\|^2 \\ =& k\|X - \frac{1}{k}\sum P_j\|^2 - \frac{1}{k}\|\sum P_j\|^2 + \sum \|P_j\|^2 \end{aligned}$$

Thus, the right-hand side is minimized when $X = \frac{1}{k}\sum P_j$. This elementary algebra computation solves both parts a) and b) of this problem.

Method 3). Because $Q(X)$ blows up as $\|X\| \to \infty$, $Q(X)$ attains its global minimum at some finite point, X_0. [Detail: There is an $R > 0$ so that if $\|X\| > R$ then $Q(X) > Q(0)$. Thus in the compact set $\|X\| \leq R$ the continuous function $Q(X)$ attains its minimum value.] This interior point must be a critical point of Q. Because from part a) we found Q has only one critical point, this must be the point X_0

B–2. Suppose that $G : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function with the property that for some real M

$$\|G(x) - G(y)\| \leq M\|x - y\| \qquad \text{for all } \; x, y \in \mathbb{R}^n. \tag{1}$$

Here $\|x\|$ is the standard Euclidean distance in $\mathbb{R}^n$.

If $\lambda > 0$ is small enough, show that the function $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F(x) = x - \lambda G(x)$$

is one-to-one and onto, so for every $z \in \mathbb{R}^n$ the equation $F(x) = z$ has one and only one solution $x \in \mathbb{R}^n$. Note that a solution x is a fixed point of some map.

SOLUTION: Rewrite the equation $F(x) = z$ as

$$x = z + \lambda G(x). \tag{2}$$

If we define the map $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(v) := z + \lambda G(v)$, then the desired solution x of (2) is a fixed point of T. We apply the Contracting Mapping Theorem to this map T on the complete metric space $\mathbb{R}^n$. We need only find a $\lambda > 0$ for which T is contracting. But for any x and y, by the inequality (1)

$$\|T(x) - T(y)\| = \|\lambda G(x) - \lambda G(y)\| \leq M\lambda\|x - y\|.$$

Thus T is contracting if $\lambda M < 1$, that is, $\lambda < 1/M$.

B–3. Compute $J = \iint_{\mathbb{R}^2} \frac{1}{(1+4x^2+5y^2)^3}\,dx\,dy\,.$

Solution: First make the change of variable $u = 2x$ and $v = \sqrt{5}y$ so $du\,dv = 2\sqrt{5}\,dx\,dy$ and

$$J = \frac{1}{2\sqrt{5}} \iint_{\mathbb{R}^2} \frac{1}{(1+u^2+v^2)^3}\,du\,dv.$$

Now in the uv plane use polar coordinates to find

$$J = \frac{1}{2\sqrt{5}} \int_0^{2\pi} \left(\int_0^{\infty} \frac{1}{(1+r^2)^3}\,r dr \right) d\theta = \frac{2\pi}{2\sqrt{5}} \int_0^{\infty} \frac{r\,dr}{(1+r^2)^3}.$$

Finally, the change of variable $s = 1 + r^2$ completes the job:

$$J = \frac{\pi}{2\sqrt{5}} \int_1^{\infty} s^{-3}\,ds = \frac{\pi}{4\sqrt{5}}.$$

B–4. Let $y = f(x,u)$ and $z = g(x,u,v)$ be smooth functions with, say, $f(x_0,u_0) = y_0$ and $g(x_0,u_0,v_0) = z_0\,.$

a) Under what condition(s) can one eliminate x from the first of these equations to express z as a smooth function of y, u, and v near $y = y_0\,,\ u = u_0\,,\ v = v_0$?

Solution: If $f_x(x_0,u_0) \neq 0$, then we can solve $y = f(x,u)$ for x as a function of y and u, which we write as $x = x(y,u)$. Then

$$z = g(x(y,u),\,u,\,v)$$

is the desired function.

b) Assuming this, then compute $\partial z/\partial u$ in terms of the derivatives of f and g. To make this computation more specific, assume that

$$f_x(x_0,u_0) = 1, \quad f_u(x_0,u_0) = -2, \quad g_x(x_0,u_0,v_0) = -3, \quad g_u(x_0,u_0,v_0) = 4, \text{ and } g_v(x_0,u_0,v_0) = -2.$$

Solution: We compute $\partial z/\partial u$ using the chain rule:

$$\frac{\partial z}{\partial u} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial u}.$$

Also, taking the partial derivative of $f(x(y,u),\,u) - y = 0$ with respect to u we find $f_x x_u + f_u = 0$ so $x_u = -f_u/f_x$ (here is where we use that $f_x \neq 0$). Using the numerical values specified, $x_u = 2/1 = 2$. Consequently $z_u = (-3)(2) + 4 = -2$.

B–5. Let $0 < b < a$. In class we parametrized the standard torus (surface of a doughnut) in $\mathbb{R}^3$ as $T : (\theta,\phi) \mapsto (x,y,z)$ where

$$x = (a + b\cos\phi)\cos\theta, \qquad y = (a + b\cos\phi)\sin\theta, \qquad z = b\sin\phi, \qquad 0 \le \theta < 2\pi,\ 0 \le \phi < 2\pi.$$

Let $\theta_0 = 0$ and $\phi_0 = \pi/2$.

a) Compute $T(\theta_0, \phi_0)$ and $DT(\theta_0, \phi_0)$.

SOLUTION:

$$T(0, \pi/2) = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} \quad \text{and} \quad DT(\theta, \phi) = \begin{pmatrix} -(a + b\cos\phi)\sin\theta & -b\sin\phi\cos\theta \\ (a + b\cos\phi)\cos\theta & -b\sin\phi\sin\theta \\ 0 & b\cos\phi \end{pmatrix}$$

so $DT(0, \pi/2) = \begin{pmatrix} 0 & -b \\ a & 0 \\ 0 & 0 \end{pmatrix}$.

b) Find the equation of the tangent plane (in $\mathbb{R}^3$) at the point $T(\theta_0, \phi_0)$.

SOLUTION: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} + \begin{pmatrix} 0 & -b \\ a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \theta - 0 \\ \phi - \frac{\pi}{2} \end{pmatrix}$

Thus $x = a - b(\phi - \frac{\pi}{2}), \quad y = a\theta, \quad z = b$.

Because here the parametera θ and ϕ can be any real numbers, we can simply describe this as the plane $z = b$.

[Last revised December 28, 2015]