

DIRECTIONS This exam has two parts. Part A has 6 shorter questions (7 points each so 42 points) while Part B had 5 problems (15 points each, so 75 points for this part). Maximum total score is thus 117 points.

Closed book, no calculators etc. – but you may use one 3" × 5" card with notes on both sides.

Remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 10:30 and ends at 12:00.

Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: There are 6 short answer questions, 7 points each so 42 points for this part.

A-1. Give an example of a sequence of smooth functions  $f_k : \mathbb{R} \rightarrow \mathbb{R}$  so that the infinite series  $\sum_{k=1}^{\infty} f_k(x)$  converges uniformly for all real  $x$  but the series  $\sum_{k=1}^{\infty} f'_k(x)$  diverges at  $x = 0$ .

A-2. If the radius of convergence of the real power series  $\sum_{n=0}^{\infty} a_n x^n$  is  $R < \infty$ , then  $a_n R^n \rightarrow 0$ .  
Proof or counterexample.

A-3. Let  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$  be a smooth curve in  $\mathbb{R}^3$  parameterized by arc length  $s$ , so  $\|\gamma'(s)\| = 1$ .  
Show that the vector  $\gamma''(s)$  is perpendicular to the tangent vector,  $\gamma'(s)$ .

A-4. Let  $f_k : \mathbb{R} \rightarrow \mathbb{R}$  be a sequence of continuous functions. Assume that  $f_k(x) = 0$  for all  $x \geq k$  so the integrals  $\int_0^{\infty} f_k(x) dx$  all exist. If  $f_k$  converges uniformly to 0 on the set  $x \geq 0$ , does  $\int_0^{\infty} f_k(x) dx$  converge to 0? Proof or counterexample.

A-5. The *Gamma function* is defined by the improper integral  $\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} dx$ . For which real values of  $t$  does this improper integral exist? Why?

A-6. Must the boundary of a set of measure zero have measure zero? Proof or counterexample.

PART B 5 questions, 15 points each (so 75 points total).

B-1. Let  $P_1, P_2, \dots, P_k$  be distinct points in  $\mathbb{R}^n$ .

a) Find the point  $X_0 \in \mathbb{R}^n$  that minimizes the function

$$Q(X) = \|X - P_1\|^2 + \|X - P_2\|^2 + \dots + \|X - P_k\|^2.$$

b) Why is the point  $X_0$  you just found the *global* minimum of  $Q(X)$ ? That is, why is  $Q(X_0) \leq Q(X)$  for all  $X \in \mathbb{R}^n$ ? [There are several completely different ways to show this.]

B-2. Suppose that  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function with the property that for some real  $M$

$$\|G(x) - G(y)\| \leq M\|x - y\| \quad \text{for all } x, y \in \mathbb{R}^n. \quad (1)$$

Here  $\|x\|$  is the standard Euclidean distance in  $\mathbb{R}^n$ .

If  $\lambda > 0$  is small enough, show that the function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by

$$F(x) = x - \lambda G(x)$$

is one-to-one and onto, so for every  $z \in \mathbb{R}^n$  the equation  $F(x) = z$  has one and only one solution  $x \in \mathbb{R}^n$ . Note that a solution  $x$  is a fixed point of some map.

B-3. Compute  $J = \iint_{\mathbb{R}^2} \frac{1}{[1 + 4x^2 + 5y^2]^3} dx dy$ .

B-4. Let  $y = f(x, u)$  and  $z = g(x, u, v)$  be smooth functions with, say,  $f(x_0, u_0) = y_0$  and  $g(x_0, u_0, v_0) = z_0$ .

- Under what condition(s) can one eliminate  $x$  from the first of these equations to express  $z$  as a smooth function of  $y$ ,  $u$ , and  $v$  near  $y = y_0$ ,  $u = u_0$ ,  $v = v_0$  ?
- Assuming this, then compute  $\partial z / \partial u$  in terms of the derivatives of  $f$  and  $g$ . To make this computation more specific, assume that

$$f_x(x_0, u_0) = 1, \quad f_u(x_0, u_0) = -2, \quad g_x(x_0, u_0, v_0) = -3, \quad g_u(x_0, u_0, v_0) = 4, \quad \text{and } g_v(x_0, u_0, v_0) = -2.$$

B-5. Let  $0 < b < a$ . In class we parametrized the standard torus (surface of a doughnut) in  $\mathbb{R}^3$  as  $T : (\theta, \phi) \mapsto (x, y, z)$  where

$$x = (a + b \cos \phi) \cos \theta, \quad y = (a + b \cos \phi) \sin \theta, \quad z = b \sin \phi, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi < 2\pi.$$

Let  $\theta_0 = 0$  and  $\phi_0 = \pi/2$ .

- Compute  $T(\theta_0, \phi_0)$  and  $DT(\theta_0, \phi_0)$ .
- Find the equation of the tangent plane (in  $\mathbb{R}^3$ ) at the point  $T(\theta_0, \phi_0)$ .

[Last revised December 21, 2015]