Math 361 — November 10, 2015

Exam 2

Jerry Kazdan — 10:30 – 11:50

Directions This exam has two parts. Part A has 5 shorter questions (7 points each so 35 points) while Part B had 5 problems (15 points each, so 75 points for this part). Maximum total score is thus 110 points.
Closed book, no calculators etc. – but you may use one $3'' \times 5''$ card with notes on both sides.

Remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 10:30 and ends at 11:50.
Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

Part A: There are 5 short answer questions, 7 points each so 35 points for this part.

A–1. Give an example of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ that has a critical point at the origin with the second derivative matrix $f''(x) = \left(\dfrac{\partial^2 f(x)}{\partial x_i \partial x_j}\right)$ positive semi-definite at the origin, but the origin is *not* a local minimum.

Solution: Example 1: $f(x,y) = x^2 + y^3$ Example 2: $f(x,y) = x^2 - y^4$
Example 3: $f(x,y) = x^3 + y^3$ Example 4: $f(x,y) = -x^4 - y^4$.

A–2. Explicitly find a smooth invertible map $T(x,y) = (u(x,y), v(x,y))$ from the whole plane $\mathbb{R}^2$ onto the half plane $\{(u,\, v) \in \mathbb{R}^2 \mid u > 2\}$.

Solution: Example 1: $u(x,y) = e^x + 2$, $v(x,y) = y$.
Example 2: $u(x,y) = e^{3x} + 2$, $v(x,y) = -5y + 7$.

A–3. Find the critical points of $f(x,y,z) = x^3 - 3x + y^2 + z^2$ and classify them (max, min, saddle).

Solution: $f_x = 3x^2 - 3$, $f_y = 2y$, $f_z = 2z$ so at a critical point, $x = \pm 1$, $y = 0$, $z = 0$
The critical points are therefore $P = (-1,0,0)$ and $Q = (1,0,0)$.

The second derivative matrix is

$$f''(x,y,z) = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

so

$$f''(P) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad f''(Q) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Thus P is a non-degenerate saddle point while Q is a non-degenerate local minimum.

A–4. Consider the curve described by the equation $x^2+cx+y+\sin(xy)=0$. For which value(s) of the constant c can this equation be written in the form $x=g(y)$ in a neighborhood of $(0,0)$?

SOLUTION: Writing the equation as $f(x,y)=0$, we want $g(y)$ to satisfy $f(g(y),\,y)=0$. The Implicit Function Theores states that we can find g if $f_x(0,0)\neq 0$. But $f_x=2x+c+y\cos(xy)$ so $f_x(0,0)=c$. Thus we can find g for any $c\neq 0$.

A–5. In the plane $\mathbb{R}^2$ let $f(x,y)=3$ for all points in the rectangle $0\le x\le 2,\quad 0\le y\le 1$ *except* on the line $x=1$ where $f(1,y)=7$. Is f Riemann integrable in this rectangle? Why?

SOLUTION: Partition the rectangle by n smaller rectangles of width h and height 1, so $nh=2$. The upper and lower Riemann sums will be the same except for the rectangles that contain the line $x=1$. There are at most 2 such rectangles. For these the total difference between the upper and lower Riemann sum is at most $2(7-3)h=8(2/n)=16/n$. By picking n large this can be made as small as we wish.

PART B 5 questions, 15 points each (so 75 points total).

B–1. Find the extrema of $f(x,y)=3x+2y$ subject to the constraint $3x^2+y^2=28$.

SOLUTION: We use Lagrange multipliers with $F(x,y,\lambda)=3x+2y-\lambda(3x^2+y^2-28)$, Thus we solve

$$0=F_x=3-6\lambda x \qquad \text{and} \qquad 0=F_y=2-2\lambda y$$

along with the constraint equation $3x^2+y^2=28$. Eliminating λ from the first two equations we find $y=2x$. Substituting this into the constraint gives $3x^2+4x^2=28$. Thus $x^2=4$ so $x=\pm 2$ and hence $y=\pm 4$. The extrema are at the points $\pm(2,\,4)$ where $f(2,4)=14$ and $f(-2,-4)=-14$.

B–2. Let $u(x,y)$ be a smooth functions of the real variables x and y.

a) If $u(x,y)$ satisfies $4u_{xx}+3u_{yy}+2u_x-5u_y-3u=0$, show that it cannot have a local positive maximum (that is, a local maximum where the function is positive).

SOLUTION: At a local maximum, $u_x=0,\quad u_y$=0, and also $u_{xx}\le 0$ and $u_{yy}\le 0$. If at that point $u>0$, then $4u_{xx}+3u_{yy}+2u_x-5u_y-3u<0$, which contradicts the differential equation.

Also show that u cannot have a negative local minimum.

If u has a negative min, then the function $v(x,y)=-u(x,y)$ has a positive local max. But v satisfies the same equation, so this is impossible. [It is equally simple to prove this directly].

b) If a function $u(x,y)$ satisfies $4u_{xx}+3u_{yy}+2u_x-5u_y-3u=0$ in a bounded region $\mathcal{D}\in\mathbb{R}^2$ and is zero on the boundary of the region, show that $u(x,y)$ is zero throughout the region.

SOLUTION: The closure, $\bar{D}$ of D is a compact set so on $\bar{D}$ a solution u must attain its maximum and minimum somewhere. If u is not identically zero, say it is positive somewhere. Since u is zero on the boundary, it must have a positive local maximum at an interior point of D. This contradicts part a). There is a similar contradiction if u is negative somewhere.

B–3. Let $f(x,y,z) = x^2y + e^x + z$ and note that $f(0,1,-1) = 0$. Show that near $y = 1$, $z = -1$ there is a smooth function $x = g(y,z)$ with $g(1,-1) = 0$ so that $f(g(y,z),\, y,\, z) = 0$.

SOLUTION: We want $g(y,z)$ so that $f(g(y,z),\, y,\, z) = 0$. By the Implicit Function Theorem, we can find g if $f_x(0,1,-1) \neq 0$. Because $f_x = 2xy + e^x$, then $f_x(0,1,-1) = 1 \neq 0$. Thus there is such a $g(y,z)$.

Also, compute the gradient of $g(y,z)$ at $(1,-1)$.

SOLUTION: To compute the gradient of g, take the partial derivative of $f(g(y,z),\, y,\, z) = 0$ with respect to both y and z gives, respetively, to obtain

$$f_x g_y + f_y = 0. \quad \text{and} \quad f_x g_z + f_z = 0.$$

At $x = 0$, $y = 1$, $z = -1$ we have $f_x(0,1,-1) = 1$, $f_y(0,1,-1) = 0$, and $f_z(0,1,-1) = 1$. Therefore $g_y(1,-1) = 0$ and $g_z(1,-1) = -1$ so $\operatorname{grad} g(1,-1) = (0,-1)$.

B–4. Using Riemann sums, compute $\displaystyle\int_0^b \cos x\, dx$. You may use without proof that:

$$\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{-\sin\frac{1}{2}\theta + \sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}.$$

SOLUTION: For any Riemann integrable function, using a partition $a = x_0 < x_1 < \cdots < x_n = b$ with n equally space intervals of width $h = (b-a)/n$, we have

$$\int_a^b f(x)\, dx \approx f(x_1)h + f(x_2)h + \cdots + f(x_n)h,$$

where x_j can be any point in the j^{th} interval. As $n \to \infty$ this approximation gives equality.

Applied to this problem and evaluating the cosine at the right-hand end points we get

$$\int_0^b \cos x\, dx \approx [\cos h + \cos 2h + \cdots + \cos nh]h = \left[\frac{-\sin\frac{1}{2}h + \sin(n+\frac{1}{2})h}{2\sin\frac{1}{2}h}\right] h.$$

Using $\lim_{t\to 0} \sin t/t = 1$, we find $h/[2\sin\frac{1}{2}h] \to 1$. Because $nh = b$, letting $h \to 0$ we see that $[-\sin\frac{1}{2}h + \sin(n+\frac{1}{2})h] \to \sin b$. Therefore

$$\int_0^b \cos x\, dx = \sin b.$$

B–5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map $T : (u, v) \mapsto (x(u, v),\, y(u, v))$. Assume that it is an invertible map from the set A to the set B, and that $DT(u, v)$ is invertible at every point of A. Let $f(x, y)$ be a smooth function for all points $(x, y) \in B$ and let

$$g(u, v) = f(x(u, v),\, y(u, v))$$

Show that the point $x = p$, $y = q$ in B is a critical point of a (smooth) function $f(x, y)$ if and only if (p, q) is the image of critical point of $(a,\, b)$ of $g(u, v)$. [Explicitly note where you use that $DT(u, v)$ is invertible.]

SOLUTION: Note $DT = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$. Since $g = f \circ T$, by the chain rule

$$\begin{aligned} Dg =& (g_u,\, g_v) = (f_x x_u + f_y y_u,\; f_x x_v + f_y y_v) \\ =& \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = Df\, DT \end{aligned}$$

Because the square matrix DT is assumed to be invertible, $Df = 0$ if and only if $Dg = 0$. In other words T gives an exact correspondence between the critical points of f and the critical points of g.

This same proof works in any number of dimensions.

[Last revised November 19, 2015]