*My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this exam.*

*Signature* PRINTED NAME

Math 361 **Exam 2** Jerry Kazdan
November 10, 2015 10:30 – 12:00

DIRECTIONS This exam has two parts. Part A has 5 shorter questions (7 points each so 35 points) while Part B had 5 problems (15 points each, so 75 points for this part). Maximum total score is thus 110 points.
Closed book, no calculators etc. – but you may use one $3'' \times 5''$ card with notes on both sides.

Remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 10:30 and ends at 12:00.
Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: There are 5 short answer questions, 7 points each so 35 points for this part.

A–1. Give an example of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ that has a critical point at the origin with the second derivative matrix $f''(x) = \left(\dfrac{\partial^2 f(x)}{\partial x_i \partial x_j}\right)$ positive semi-definite at the origin, but the origin is *not* a local minimum.

| *Score* | |
|---|---|
| A-1 | |
| A-2 | |
| A-3 | |
| A-4 | |
| A-5 | |
| B-1 | |
| B-2 | |
| B-3 | |
| B-4 | |
| B-5 | |
| *Total* | |

A–2. Explicitly find a smooth invertible map $T(x, y) = (u(x, y), v(x, y))$ from the whole plane $\mathbb{R}^2$ onto the half plane $\{(u,\, v) \in \mathbb{R}^2 \,|\, u > 2\}$.

A–3. Find the critical points of $f(x, y, z) = x^3 - 3x + y^2 + z^2$ and classify them (max, min, saddle).

A–4. Consider the curve described by the equation $x^2 + cx + y + \sin(xy) = 0$. For which value(s) of the constant $c$ can this equation be written in the form $x = g(y)$ in a neighborhood of $(0, 0)$?

A–5. In the plane $\mathbb{R}^2$ let $f(x,y)=3$ for all points in the rectangle $0\le x\le 2$, $0\le y\le 1$ *except* on the line $x=1$ where $f(1,y)=7$. Is $f$ Riemann integrable in this rectangle? Why?

PART B 5 questions, 15 points each (so 75 points total).

B–1. Find the extrema of $f(x,y)=3x+2y$ subject to the constraint $3x^2+y^2=28$.

B–2. Let $u(x,y)$ be a smooth functions of the real variables $x$ and $y$.

a) If $u(x,y)$ satisfies $4u_{xx} + 3u_{yy} + 2u_x - 5u_y - 3u = 0$, show that it cannot have a local positive maximum (that is, a local maximum where the function is positive).

Also show that $u$ cannot have a negative local minimum.

b) If a function $u(x,y)$ satisfies $4u_{xx} + 3u_{yy} + 2u_x - 5u_y - 3u = 0$ in a bounded region $\mathcal{D} \in \mathbb{R}^2$ and is zero on the boundary of the region, show that $u(x,y)$ is zero throughout the region.

B–3. Let $f(x,y,z) = x^2y + e^x + z$ and note that $f(0,1,-1) = 0$. Show that near $y = 1$, $z = -1$ there is a smooth function $x = g(y,z)$ with $g(1,-1) = 0$ so that $f(g(y,z),\, y,\, z) = 0$.

Also, compute the gradient of $g(y,z)$ at $(1,-1)$.

B–4. Using Riemann sums, compute $\int_0^b \cos x\,dx$. You may use without proof that:

$$\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{-\sin\frac{1}{2}\theta + \sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}.$$

B–5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map $T : (u, v) \mapsto (x(u, v),\, y(u, v))$. Assume that it is an invertible map from the set $A$ to the set $B$, and that $DT(u, v)$ is invertible at every point of $A$. Let $f(x, y)$ be a smooth function for all points $(x, y) \in B$ and let

$$g(u, v) = f(x(u, v),\, y(u, v))$$

Show that the point $x = p$, $y = q$ in $B$ is a critical point of a (smooth) function $f(x, y)$ if and only if $(p, q)$ is the image of critical point of $(a,\, b)$ of $g(u, v)$. [Explicitly note where you use that $DT(u, v)$ is invertible.]