# Exam 1

DIRECTIONS This exam has two parts. Part A has ten shorter questions (10 points each so 100 points) while Part B had two problems (25 points each, so 50 points for this part). Maximum total score is thus 150 points.
Closed book, no calculators etc. – but you may use one $3'' \times 5''$ card with notes on both sides.

Remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 10:30 and ends at 12:00.
lease indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: There are ten short answer questions, 10 points each so 100 points for this part.

A–1. Let $X$ and $Y$ be linear spaces and $L : X \to Y$ be a linear map. Say $x_1$ and $x_2$ are *distinct* solutions of the equation $Lx = y$ while $x_3$ is a solution of $Lx = z$. Answer the following in terms of $x_1$, $x_2$, and $x_3$.

a) Find some solution of $Lx = y - 2z$.

SOLUTION: $x = x_1 - 2x_3$.

b) Show that the equation $Lx = z$ has infinitely many distinct solutions.

SOLUTION: Since $L(x_1 - x_2) = 0$, then for any constant $c$ let

$$x = x_3 + c(x_1 - x_2).$$

A–2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 0$ for all $x \le 0$, $f(x) > 0$ for all $x > 0$, $f \in C^2(\mathbb{R})$ but *not* $C^3$ at the origin.

SOLUTION: For $x \ge 0$ let $f(x) = x^3$. Then for $x \ge 0$ we have $f'(x) = 3x^2$, $f''(x) = 6x$, and $f'''(x) = 6$. But for $x \le 0$, $f'''(x) = 0$. These do not agree at $x = 0$.

A–3. Give an example of an infinite series $\sum_{n=1}^{\infty} f_n(x)$ where the functions $f_n : \mathbb{R} \to \mathbb{R}$ are smooth, where the series converges absolutely and uniformly for all $x \in \mathbb{R}$, yet the the differentiated series, $\sum_{n=1}^{\infty} f_n'(x)$ *diverges* at $x = 0$.

SOLUTION: Let $f_n(x) = \frac{\sin(n3x)}{n^2}$. Then $f_n'(x) = n\cos(n^3x)$. Because $|f_n(x)| \le 1/n^2$, by the Weierstrass M-Test the series $\sum f_n(x)$ converges absolutely and uniformly for all $x$. However, $\sum f_n'(0) = \sum n$ diverges.

A–4. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ and suppose there is a constant $M$ so that $\|f(x)\| \le M\|x\|^2$ for all $x \in \mathbb{R}^n$. Prove that $f$ is differentiable at the origin, $x_0 = 0$, and that $Df(0) = 0$.

SOLUTION: In the definition of the derivative as a linear map, we need to suspect in advance what the derivative is. In this case, we suspect that $Df(0) = 0$. Now for this function $f(0) = 0$ so we investigate:

$$\frac{\|f(x) - [f(0) + Df(0)]\|}{\|x\|} = \frac{\|f(x)\|}{\|x\|} \le \frac{M\|x\|^2}{\|x\|} = M\|x\|.$$

This clearly converges to zero as $x \to 0$.

A–5. At the point $(x, y) \in \mathbb{R}^2$, in what direction does the function $f(x, y) = ye^{x^2}$ increase the fastest?

SOLUTION: The direction $V$ of fastest increase if the direction of the gradient vector. Since $f_x = 2xye^{x^2}$ and $f_y = e^{x^2}$,

$$V = (2xye^{x^2},\, e^{x^2}).$$

Often, it is more useful to have a *unit* vector $V$. Here it is $V = (2xy,\, 1)/\sqrt{4x^2y^2 + 1}$.

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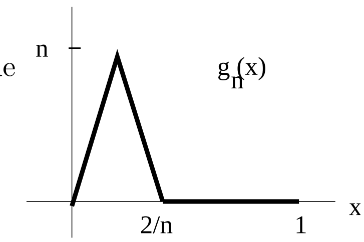
*For problems A-6 and A-7 give a proof or a counterexample to the questions. For a counterexample a clear rough sketch is adequate.*

A–6. If $f_n(x) \in C([0,\, 2])$ converge to zero pointwise, is it true that $\int_0^2 f_n(x)\, dx \to 0$?

SOLUTION: This is the graph of one counterexample.




A–7. Same as the previous problem except now assume that $f_n(x) \to 0$ uniformly in $C([0, 2])$.

SOLUTION: Given $\epsilon > 0$, pick $N$ so that if $n > N$, then $|f_n(x)| < \epsilon$ for all $x \in [0,\, 2]$. Consequently

$$\left| \int_0^2 f_n(x)\, dx \right| \le \int_0^2 |f_n(x)|\, dx \le \epsilon \int_0^2 dx = 2\epsilon.$$

Done.

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A–8. Let $f(s) : \mathbb{R} \to \mathbb{R}$ be any smooth function and let $u(x, t) := f(x + ct)$. For which real numbers $c$ does $u$ satisfy the wave equation $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}$ ?

SOLUTION: By computations using the chain rule, $u_x = f'(x + ct)$, $u_{xx} = f''(x + ct)$, $u_t = cf'(x + ct)$, and $u_{tt} = c^2 f''(x + ct)$. Thus we want

$$c^2 f''(x + ct) = 4f''(x + ct)$$

so $c^2 = 4$, that is, $c = \pm 2$.

A–9. Let $A(t)$ and $B(t)$ be $n \times n$ matrices which are differentiable functions of $t \in \mathbb{R}$. Using the definition of the derivative as the limit of a quotient, show that their product, $C(t) := A(t)B(t)$ is also differentiable as a function of $t$ and find a formula for the derivative.

SOLUTION: We need to understand the limit as $h \to 0$ of $\frac{C(t+h)-C(t)}{h}$. Note that

$$\begin{aligned}\frac{C(t+h)-C(t)}{h} &= \frac{A(t+h)B(t+h)-A(t)B(t)}{h}\\ &= \frac{[A(t+h)-A(t)]B(t+h)+A(t)[B(t+h)-B(t)]}{h}\\ &= \frac{A(t+h)-A(t)}{h}B(t+h)+A(t)\frac{B(t+h)-B(t)}{h}\end{aligned}$$

Since both $A$ and $B$ are given to be differentiable, we can let $h \to 0$ to find

$$C'(t) = A'(t)B(t) + A(t)B'(t).$$

[NOTE: One needs to be a bit careful *not* to assume that $AB = BA$.]

A–10. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be the map $F(x,y,z) = \begin{pmatrix} xy - 2z + 3 \\ e^{z\cos(2y)} \end{pmatrix}$. Compute $DF$ at the point $(2,0,1)$.

SOLUTION: By a direct computation:

$$Df(x,y,z) = \begin{pmatrix} y & x & -2 \\ 0 & -2z\sin(2y)e^{z\cos(2y)} & \cos(2y)e^{z\cos(2y)} \end{pmatrix}.$$

In particular

$$Df(2,0,1) = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 0 & e \end{pmatrix}.$$

PART B Two questions, 25 points each (so 50 points total).

B–1. Let $g_n : \mathbb{R} \to \mathbb{R}$ be continuous functions with the following three properties:

$$(i)\ g_n(x) \ge 0 \ \text{ for } \ |x| < \frac{1}{n}, \qquad (ii)\ g_n(x) = 0 \ \text{ for } \ |x| \ge \frac{1}{n}, \qquad (iii)\ \int_{-\infty}^{\infty} g_n(x)\,dx = 1.$$

Let $f(x) : \mathbb{R} \to \mathbb{R}$ be continuous for all $x$ with $f(x) = 0$ for $|x| \ge 1$. Define

$$h_n(t) = \int_{-\infty}^{\infty} f(t-x)g_n(x)dx.$$

a) Show that $h_n$ is uniformly continuous.

SOLUTION: Since $f$ is continuous for all $x$ and zero outside of a compact set, it is uniformly continuous on $\mathbb{R}$. Thus, given $\epsilon > 0$ there is a $\delta > 0$ so that if $|x_2 - x_1| < \delta$, then $|f(x_2) - f(x_1)| < \epsilon$. Now

$$h(t_2) - h(t_1) = \int_{-\infty}^{\infty} [f(t_2 - x) - f(t_1 - x)g_n(x)\,dx.$$

However, if $|t_2 - t_1| < \delta$, then $|(t_2 - x) - (t_1 - x)| = |t_2 - t_1| < \delta$. Therefore $|f(t_2 - x) - f(t_1 - x)| < \epsilon$ so

$$|h_n(t_2) - h_n(t_1)| \le \int_{-\infty}^{\infty} \epsilon g_n(x)\,dx = \epsilon.$$

This shows that $h_n$ is uniformly continuous.

b) Show that $\lim_{n\to\infty} h_n(t) = f(t)$ *uniformly.*

SOLUTION: Begin with the natural (but not completely trivial)

$$h_n(t) - f(t) = \int_{-\infty}^{\infty} f(t-x)g_n(x)\,dx - f(t)\int_{-\infty}^{\infty} g_n(x)\,dx = \int_{-1/n}^{1/n}[f(t-x) - f(t)]g_n(x)\,dx.$$

Note that here $|x| < 1/n$. Given $\epsilon > 0$, using $\delta$ from part a), pick $N$ so that $1/N < \delta$. Then, as in part a), $|f(t-x) - f(t)| < \epsilon$. Consequently, if $n > N$ then

$$|h_n(t) - f(t)| < \epsilon \int_{-1/n}^{1/n} g_n(x)\,dx = \epsilon.$$

Because this holds for all $t$, it proves the desired uniform convergence.

B–2. Let $K(x, y)$ be continuous for $x \in [0, 1]$ and $y \in [0, 1]$ and assume that $|K(x, y)| \le 3$ for these $(x, y)$. Find an explicit constant $\lambda > 0$ so that for any $f \in C([0,\,1])$ the integral equation

$$u(x) = f(x) + \lambda \int_0^1 K(x, y)u(y)\,dy$$

has an unique solution $u \in C([0,\,1])$.

SOLUTION: We will use a contracting mapping argument with the complete metric space $C([0, 1])$ of continuous functions in the interval $0 \le x \le 1$ with the standard uniform norm: $\|u\| = \max_{0\le x\le 1}|u(x)|$.

Define the map

$$Tv(x) := f(x) + \lambda \int_0^1 K(x, y)v(y)\,dy.$$

The desired solution $u(x)$ will be a fixed point of $T$. Since $f(x)$ and $K(x, y)$ are continuous functions of $x$, clearly $T : C([0, 1])$ to itself. We only need to pick $\lambda > 0$ so that $T$ is a contracting map. But

$$\begin{aligned}|Tv(x) - Tw(x)| =& \lambda \left| \int_0^1 K(x, y)[v(y) - w(y)]\,dy \right| \\ \le& 3\lambda \int_0^1 |v(y) - w(y)|\,dy \le 3\lambda\|v - w\|.\end{aligned}$$

Since the right side is independent of $x$, we can take the maximum of the left side over all $x \in [0, 1]$ to conclude that

$$\|Tv - Tw\| \le 3\lambda\|v - w\|.$$

Then $T$ will be contracting if we pick any $\lambda < 1/3$, say $\lambda = 1/6$.

[Last revised October 6, 2015]