Code of Academic Integrity in completing this exam.

Signature PRINTED NAME

Math 361 | **Exam 1** | Jerry Kazdan

October 1, 2015 | | 10:30 – 12:00

DIRECTIONS This exam has two parts. Part A has ten shorter questions (10 points each so 100 points) while Part B had two problems (25 points each, so 50 points for this part). Maximum total score is thus 150 points.

Closed book, no calculators etc. – but you may use one $3'' \times 5''$ card with notes on both sides.

Remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 10:30 and ends at 12:00.

Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: There are ten short answer questions, 10 points each so 100 points for this part.

A–1. Let X and Y be linear spaces and $L : X \to Y$ be a linear map. Say x_1 and x_2 are *distinct* solutions of the equation $Lx = y$ while x_3 is a solution of $Lx = z$. Answer the following in terms of x_1, x_2, and x_3.

a) Find some solution of $Lx = y - 2z$.

b) Show that the equation $Lx = z$ has infinitely many distinct solutions.

A–2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 0$ for all $x \le 0$, $f(x) > 0$ for all $x > 0$, $f \in C^2(\mathbb{R})$ but *not* C^3 at the origin.

Score	
A-1	
A-2	
A-3	
A-4	
A-5	
A-6	
A-7	
A-8	
A-9	
A-10	
B-1	
B-2	
Total	

A–3. Give an example of an infinite series $\sum_{n=1}^{\infty} f_n(x)$ where the functions $f_n : \mathbb{R} \to \mathbb{R}$ are smooth, where the series converges absolutely and uniformly for all $x \in \mathbb{R}$, yet the the differentiated series, $\sum_{n=1}^{\infty} f'_n(x)$ *diverges* at $x = 0$.

A–4. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ and suppose there is a constant M so that $\|f(x)\| \leq M\|x\|^2$ for all $x \in \mathbb{R}^n$. Prove that f is differentiable at the origin, $x_0 = 0$, and that $Df(0) = 0$.

A–5. At the point $(x, y) \in \mathbb{R}^2$, in what direction does the function $f(x, y) = ye^{x^2}$ increase the fastest?

For problems A-6 and A-7 give a proof or a counterexample to the questions. For a counterexample a clear rough sketch is adequate.

A–6. If $f_n(x) \in C([0,\ 2])$ converge to zero pointwise, is it true that $\int_0^2 f_n(x)\,dx \to 0$?

A–7. Same as the previous problem except now assume that $f_n(x) \to 0$ uniformly in $C([0,2])$.

A–8. Let $f(s) : \mathbb{R} \to \mathbb{R}$ be any smooth function and let $u(x,t) := f(x+ct)$. For which real numbers c does u satisfy the wave equation $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}$?

A–9. Let $A(t)$ and $B(t)$ be $n \times n$ matrices which are differentiable functions of $t \in \mathbb{R}$. Using the definition of the derivative as the limit of a quotient, show that their product, $C(t) := A(t)B(t)$ is also differentiable as a function of t and find a formula for the derivative.

A–10. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be the map $F(x, y, z) = \begin{pmatrix} xy - 2z + 3 \\ e^{z \cos 2y} \end{pmatrix}$. Compute DF at the point $(2, 0, 1)$.

PART B Two questions, 25 points each (so 50 points total).

B–1. Let $g_n : \mathbb{R} \to \mathbb{R}$ be continuous functions with the following three properties:

$$(i)\ g_n(x) \geq 0 \ \text{ for } \ |x| < \frac{1}{n}, \qquad (ii)\ g_n(x) = 0 \ \text{ for } \ |x| \geq \frac{1}{n}, \qquad (iii)\ \int_{-\infty}^{\infty} g_n(x)\, dx = 1.$$

[REMARK: If one has $g_1(x)$, then defining $g_n(x) := n g_1(nx)$ gives a sequence of g_n's with the above properties.]

Let $f(x) : \mathbb{R} \to \mathbb{R}$ be continuous for all x with $f(x) = 0$ for $|x| \geq 1$. Define

$$h_n(t) = \int_{-\infty}^{\infty} f(t-x) g_n(x) dx.$$

a) Show that h_n is uniformly continuous.

b) Show that $\lim_{n\to\infty} h_n(t) = f(t)$ *uniformly.*

REMARK: If the $g_n(x)$ are smooth, it is not difficult to show that the $h_n(x)$ are also smooth. Thus this problem gives a way to prove that one can *uniformly* approximate any continuous function by a smooth function.

B–2. Let $K(x,y)$ be continuous for $x \in [0,1]$ and $y \in [0,1]$ and assume that $|K(x,y)| \le 3$ for these (x,y). Find an explicit constant $\lambda > 0$ so that for any $f \in C([0,\,1])$ the integral equation

$$u(x) = f(x) + \lambda \int_0^1 K(x,y)u(y)\,dy$$

has a unique solution $u \in C([0,\,1])$.

[Last revised October 3, 2015]