A Tridiagonal Matrix

We investigate the simple $n \times n$ real tridiagonal matrix:

$$M = \begin{pmatrix} \alpha & \beta & 0 & 0 & \dots & 0 & 0 \\ \beta & \alpha & \beta & 0 & \dots & 0 & 0 \\ 0 & \beta & \alpha & \beta & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha & \beta & 0 \\ 0 & 0 & 0 & \dots & \beta & \alpha & \beta \\ 0 & 0 & 0 & \dots & 0 & \beta & \alpha \end{pmatrix} = \alpha I + \beta \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} = \alpha I + \beta T,$$

where T is defined by the preceding formula. This matrix arises in many applications, such as n coupled harmonic oscillators and solving the Laplace equation numerically. Clearly M and T have the same eigenvectors and their respective eigenvalues are related by $\mu = \alpha + \beta\lambda$. Thus, to understand M it is sufficient to work with the simpler matrix T.

Eigenvalues and Eigenvectors of T

Usually one first finds the eigenvalues and then the eigenvectors of a matrix. For T, it is a bit simpler first to find the eigenvectors. Let λ be an eigenvalue (necessarily real) and $V = (v_1, v_2, \dots, v_n)$ be a corresponding eigenvector. With hindsight it will be convenient to write $\lambda = 2c$. Then

$$0 = (T - \lambda I)V = \begin{pmatrix} -2c & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -2c & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2c & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2c & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -2c & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -2c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \\ v_n \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} -2cv_1 + v_2 \\ v_1 - 2cv_2 + v_3 \\ \vdots \\ v_{k-1} - 2cv_k + v_{k+1} \\ \vdots \\ v_{n-2} - 2cv_{n-1} + v_n \\ v_{n-1} - 2cv_n \end{pmatrix}$$

Except for the first and last equation, these have the form

$$v_{k-1} - 2cv_k + v_{k+1} = 0. \tag{2}$$

We can also bring the first and last equations into this same form by introducing new artificial variables v_0 and v_{n+1}, setting their values as zero: $v_0 = 0$, $v_{n+1} = 0$.

The result (2) is a *second order linear difference equation with constant coefficients* along with the *boundary conditions* $v_0 = 0$, and $v_{n+1} = 0$. As usual for such equations one seeks a solution with the form $v_k = r^k$. Equation (2) then gives $1 - 2cr + r^2 = 0$ whose roots are

$$r_\pm = c \pm \sqrt{c^2 - 1}$$

Note also

$$2c = r + r^{-1} \qquad \text{and} \qquad r_+ r_- = 1. \tag{3}$$

Case 1: $c \neq \pm 1$. In this case the two roots $r_\pm$ are distinct. Let $r := r_+ = c + \sqrt{c^2-1}$. Since $r_- = c - \sqrt{c^2-1} = 1/r$, we deduce that the general solution of (1) is

$$v_k = A r_+^k + B r_-^k = A r^k + B r^{-k}, \qquad k = 0, \ldots, n+1 \tag{4}$$

for some constants A and B.

The first boundary condition, $v_0 = 0$, gives $A + B = 0$, so

$$v_k = A(r^k - r^{-k}), \qquad k = 0, \ldots, n+1. \tag{5}$$

Since for a non-trivial solution we need $A \neq 0$, the second boundary condition, $v_{n+1} = 0$, implies

$$r^{n+1} - r^{-(n+1)} = 0, \qquad \text{so} \qquad r^{2(n+1)} = 1.$$

In particular, $|r| = 1$. Using (3), this gives $2|c| \leq |r| + |r|^{-1} = 2$. Thus $|c| \leq 1$. In fact, $|c| < 1$ because we are assuming that $c \neq \pm 1$.

Case 2: $c = \pm 1$. Then $r = c$ and the general solution of (1) is now

$$v_k = (A + Bk)c^k.$$

The boundary condition $v_0 = 0$ implies that $A = 0$. The other boundary condition then gives $0 = v_{n+1} = B(n+1)c^{n+1}$. This is satisfied only in the trivial case $B = 0$. Consequently the equations (1) have no non-trivial solution for $c = \pm 1$.

It remains to rewrite our results in a simpler way. We are in Case 1 so $|r| = 1$. Thus $r = e^{i\theta}$, $c = \cos\theta$, and $1 = r^{2(n+1)} = e^{2i(n+1)\theta}$. Consequently $2(n+1)\theta = 2k\pi$ for some $1 \leq k \leq n$ (we exclude $k = 0$ and $k = n+1$ because we know that $c \neq \pm 1$, so $r \neq \pm 1$). Normalizing the eigenvectors V by the choice $A = 1/2i$, we summarize as follows:

Theorem 1 *The $n \times n$ matrix T has the eigenvalues*

$$\lambda_k = 2c = 2\cos\theta = 2\cos\tfrac{k\pi}{n+1}, \qquad 1 \le k \le n$$

and corresponding eigenvectors

$$V_k = (\sin\tfrac{k\pi}{n+1}, \sin\tfrac{2k\pi}{n+1}, \dots, \sin\tfrac{nk\pi}{n+1}).$$

REMARK 1. If $n = 2k+1$ is odd, then the middle eigenvalue is zero because $(k+1)\pi/(n+1) = (k+1)\pi/2(k+1) = \pi/2$.

REMARK 2. Since $2ab = a^2 + b^2 - (a-b)^2 \le a^2 + b^2$ with equality only if $a = b$, we see that for any $x \in \mathbb{R}^n$

$$\langle x, Tx\rangle = 2(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n) \le x_1^2 + 2(x_2^2 + \cdots + x_{n-1}^2) + x_n^2 \le 2\|x\|^2$$

with equality only if $x = 0$. Similarly $\langle x, Tx\rangle \ge -2\|x\|^2$. Thus, the eigenvalues of T are in the interval $-2 < \lambda < 2$. Although we obtained more precise information above, it is useful to observe that we could have deduced this so easily.

REMARK 3. *Gershgorin's circle theorem* is also a simple way to get information about the eigenvalues of a square (complex) matrix $A = (a_{ij})$. Let D_i be the disk in the complex plane whose center is at a_{ii} and radius is $R_i = \sum_{j\ne i}|a_{ij}|$, so

$$|\lambda - a_{jj}| \le R_j.$$

These are the *Gershgorin disks*.

Theorem 2 (Gershgorin) *Each eigenvalues of A lies in at least one of these Gershgorin discs.*

Proof: Say $Ax = \lambda x$ and say $|x_i| = \max_j |x_j|$. The i^{th} component of $Ax = \lambda x$ is

$$(\lambda - a_{ii})x_i = \sum_{j\ne i} a_{ij}x_j$$

so

$$|(\lambda - a_{ii})x_i| \le \sum_{j\ne i}|a_{ij}||x_j| \le R_i|x_i|.$$

That is, $|\lambda - a_{ii}| \le R_i$, as claimed.

Example By Gershgorin's theorem, we observed immediately that all of the eigenvalues of T satisfy $|\lambda| \le 2$.

Determinant of $T - \lambda I$

We use recursion on n, the size of the $n \times n$ matrix T. It will be convenient to build on (1) and let $D_n = \det(T - \lambda I)$. As before, write $\lambda = 2c$. Then, expanding by minors using the first column of (1) we obtain the formula

$$D_n = -2cD_{n-1} - D_{n-2} \qquad n = 3, 4, \dots. \tag{6}$$

Since $D_1 = -2c$ and $D_2 = 4c^2 - 1$, we can use (6) to define $D_0 := 1$. The relation (6) is, except for the sign of c, is identical to (2). The solution for $c \neq \pm 1$ is thus

$$D_k = As^k + Bs^{-k}, \qquad k = 0, 1, \dots, \tag{7}$$

where

$$-2c = s + s^{-1} \qquad \text{and} \qquad s = -c + \sqrt{c^2 - 1}. \tag{8}$$

This time we determine the constants A, B from the *initial conditions* $D_0 = 1$ and $D_1 = -2c$. The result is

$$D_k = \begin{cases} \dfrac{1}{2\sqrt{c^2-1}}\left(s^{k+1} - s^{-(k+1)}\right) & \text{if} \quad c \neq \pm 1, \\ (-c)^k (k+1) & \text{if} \quad c = \pm 1. \end{cases} \tag{9}$$

For many purposes it is useful to rewrite this.

Case 1: $|c| < 1$. Then $s = -c + i\sqrt{1 - c^2}$ has $|s| = 1$ so $s = e^{i\alpha}$ and $c = -\cos\alpha$ for some $0 < \alpha < \pi$. Therefore from (9),

$$D_k = \frac{\sin(k+1)\alpha}{\sin\alpha}. \tag{10}$$

Case 2: $c > 1$. Write $c = \cosh\beta$ for some $\beta > 0$. Since $-e^{\beta} - e^{-\beta} = -2c = s + s^{-1}$, write $s = -e^{\beta}$. Then from (9),

$$D_k = (-1)^k \frac{\sinh(k+1)\beta}{\sinh\beta}, \tag{11}$$

where we chose the sign in $\sqrt{c^2 - 1} = -\sinh\beta$ so that $D_0 = 1$.

Case 3: $c < -1$. Write $c = -\cosh\beta$ for some $\beta > 0$. Since $e^t + e^{-t} = -2c = s + s^{-1}$, write $s = e^{\beta}$. Then from (9),

$$D_k = \frac{\sinh(k+1)\beta}{\sinh\beta}, \tag{12}$$

where we chose the sign in $\sqrt{c^2 - 1} = +\sinh t$ so that $D_0 = 1$.

Note that as $t \to 0$ in (10)–(12), that is, as $c \to \pm 1$. these formulas agree with the case $c = \pm 1$ in (9).

A Vibrating String (coupled oscillators)

Say we have n particles with the same mass m equally spaced on a string having tension τ. Let y_k denote the vertical displacement if the kth mass. Assume the ends of the string are fixed; this is the same as having additional particles at the ends, but with zero displacement: $y_0 = 0$ and $y_{n+1} = 0$. Let ϕ_k be the angle the segment of the string between the kth and k+1st particle makes with the horizontal. Then Newton's second law of motion applied to the kth mass asserts that

$$m\ddot{y}_k = \tau \sin\phi_k - \tau \sin\phi_{k-1}, \qquad k = 1, \ldots, n. \tag{13}$$

If the particles have horizontal separation h, then $\tan\phi_k = (y_{k+1} - y_k)/h$. For the case of small vibrations we assume that $\phi_k \approx 0$; then $\sin\phi_k \approx \tan\phi_k = (y_{k+1} - y_k)/h$ so we can rewrite (13) as

$$\ddot{y}_k = p^2(y_{k+1} - 2y_k + y_{k-1}), \qquad k = 1, \ldots, n, \tag{14}$$

where $p^2 = \tau/mh$. This is a system of second order linear constant coefficient differential equations with the boundary conditions $y_0(t) = 0$ and $y_{n+1}(t) = 0$. As usual, one seeks special solutions of the form $y_k(t) = v_k e^{\alpha t}$. Substituting this into (14) we find

$$\alpha^2 v_k = p^2(v_{k+1} - 2v_k + v_{k-1}), \qquad k = 1, \ldots, n,$$

that is, α^2 is an eigenvalue of $p^2(T - 2I)$. From the work above we conclude that

$$\alpha_k^2 = -2p^2(1 - \cos\tfrac{k\pi}{n+1}) = -4p^2 \sin^2 \tfrac{k\pi}{2(n+1)}, \qquad k = 1, \ldots, n,$$

so

$$\alpha_k = 2ip \sin \tfrac{k\pi}{2(n+1)}, \qquad k = 1, \ldots, n.$$

The corresponding eigenvectors V_k are the same as for T. Thus the special solutions are

$$Y_k(t) = V_k e^{2ipt \sin \frac{k\pi}{2(n+1)}}, \qquad k = 1, \ldots, n,$$

where $Y(t) = (y_1(t), \ldots, y_n(t))$.