Math 320 — **Exam 1** — Jerry Kazdan

October 2, 2018 — 1:30 – 2:50

DIRECTIONS This exam has two parts. Part A has six shorter questions (5 points each so 30 points) while Part B had four problems (10 points each, so 40 points for this part). Maximum total score is thus 70 points.

Closed book, no calculators etc. – but you may use one $3'' \times 5''$ card with notes on both sides.

Remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 1:30 and ends at 2:50.

Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: There are 6 short answer questions, 5 points each so 30 points for this part.

A–1. Write 271.4 in base 2.

A–2. This concerns computing $\displaystyle\sum_{k=1}^{n}\sum_{j=1}^{k} a_k b_j$.

a) How many multiplications and additions are needed?

b) Modify the above sum to an equivalent form that reduced the number of computations.

A–3. The Taylor polynomial of degree n for e^x is $e^x \approx \sum_0^n x^k/k!$. Someone uses this Taylor polynomial of degree 9 and three digit chopping arithmetic to find an approximation to e^{-5} by both of the following methods:

a). $e^{-5} \approx \sum_0^9 \frac{(-5)^k}{k!}$ b). $e^{-5} = \dfrac{1}{e^5} \approx \dfrac{1}{\sum_0^9 \frac{5^k}{k!}}$.

Which will be more accurate? Why?

A–4. Someone uses three-digit chopping arithmetic to compute $\sum_1^{10}(1/k^2)$ in two different ways: $\frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{100}$ and $\frac{1}{100} + \cdots + \frac{1}{1}$. Which method is more accurate? Why?

A–5. Let $p(x) := (x+2)(x+1)x(x-1)(x-2)$. To which zero of p does the Bisection method converge when applied to the interval $[-0.5, 3]$?

A–6. Let $g(x) = x^2$.

a) Find the fixed points of g.

b) Which of these fixed points can be approximated using fixed point iteration $x_{k+1} = g(x_k)$? Explain your reasoning, Part B Four questions, 10 points each (so 40 points total).

B–1. Explain how you would approximate $5^{1/3}$ using Newton's method. Your response should involve an explicit iteration sequence that you suspect approximates $5^{1/3}$. [You are not being asked to do explicit decimal calculations.]

B–2. Say you have four data points (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3). Describe how you would construct a "natural" cubic spline approximation. How many cubic polynomials are needed? How can you determine their coefficients?

B–3. Consider designing a quadature rule of the form

$$\int_0^1 f(x)\,dx \approx af(0) + bf(c)$$

for some weights a, b and undetermined node $0 < c < 1$.

a) What is the maxximum degree of accuracy that this ethod can achieve? [That is, the maximum degree n so that this rule has no error for polynomials of degree n.]

b) Explicitly find a, b, c that allows the above rule to achieve this degree of accuracy.

B–4. [Zeros of polynomials] Let $p(z) := z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where the coefficients a_k may be complex numbers. Find a disk $D := \{z \in \mathbb{C} \,|\, |z| \le R\}$ so that all of the (possibly complex) zeroes of p lie in this disk. Of course the radius R depends on the coefficients.

There are several approaches. Here is one that works for cubic polynomials. It begins with the observation that if $z \ne 0$ is a root of $p(z) = z^3 + a_2 z^2 + a_1 z + a_0$, then

$$z = -\left(a_2 + \frac{a_1}{z} + \frac{a_0}{z^2}\right)$$

so if $|z| \ge 1$ then

$$|z| \le |a_2| + \left|\frac{a_1}{z}\right| + \left|\frac{a_0}{z^2}\right| \le |a_2| + |a_1| + |a_0|.$$

Thus the zeroes of $p(z)$ all lie in the disk centered at the origin whose radius is the larger of 1 and $|a_2| + |a_1| + |a_0|$. Generalize this to polynomials of degree n.

[Last revised October 3, 2018]