Math 313/515, Spring 2005 Jerry L. Kazdan

**Homework Set 9, Due Thursday, March 31, 2005**
(*Late papers will be accepted until 4 PM on Fri. April 1*)

*Some of these problems use the same matrices as Homework Set 8. That should save you some time.*

1. Let $u(t) = (u_1(t), u_2(t))$. Solve the differential equation $\dfrac{du}{dt} = Au$ with $u(0) = (1,2)$ where for $A$ you use the following matrices:

   a). $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ b). $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ c). $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ d). $\begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix}$.

2. Find the general solution of $\dfrac{du}{dt} = Au$, where $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 4 & -2 \end{pmatrix}$.

3. [Strang, p. 299 #10]. Get $G_k$ be a sequence of numbers with the property

$$G_{k+2} = \tfrac{1}{2}(G_{k+1} + G_k), \quad \text{with} \quad G_0 = a \text{ and } G_1 = b.$$

   a) Find an explicit formula for $G_k$ by diagonalizing an appropriate matrix.

   b) Compute $\lim_{k\to\infty} G_k$ in terms of $a$ and $b$. [You may find it useful to try the special case where $G_0 = 1$ and $G_1 = 3$.

4. [Strang, p. 301 #27]. Say $A = S\Lambda S^{-1}$, where $\Lambda$ is a diagonal matrix, and $B$ is the *block matrix* $B = \left(\begin{smallmatrix} A & 0 \\ 0 & 2A \end{smallmatrix}\right)$. Diagonalize $B$.

5. In Homework 7 we worked with $\Delta_n = \det M_n$ be the determinant of an $n \times n$ matrix $M_n$ with $a$'s along the main diagonal and $b$'s on the two "off diagonals" directly above and below the main diagonal (this is a simple example of a *tridiagonal* matrix). Thus

$$M_5 = \begin{pmatrix} a & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 \\ 0 & b & a & b & 0 \\ 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix}.$$

   You showed that $\Delta_n = a\Delta_{n-1} - b^2\Delta_{n-2}$.

   The task now is to find an explicit formula for $\Delta_n$

6. Let $A$ be a real $2 \times 2$ matrix with the property that $A^3 = I$.

   a) If $\lambda$ is an eigenvalue of $A$, show that $\lambda^3 = 1$.

   b) What are all possible values of the trace and determinant of $A$?

   c) Use this to all possible real matrices $A$ satisfying $A^3 = I$.

7. If $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = I + A$, compute $A^2$, $A^3$, $e^A$, and $B^2$, $B^3$, $e^B$.

8. Let $B$ be a real antisymmetric matrix. Show that $M := e^B$ is an orthogonal matrix.

9. Let $M$ be a diagonalizable real $n \times n$ matrix with (possible complex) eigenvalues $\lambda_1$, $\lambda_2, \ldots, \lambda_n$. If the real parts of these eigenvalues are all *negative*, show that $e^{Mt} \to 0$ as $t \to \infty$.

10. Let $A$ be a real square matrix. If If $\lambda$ is a real eigenvalue of $A$ with corresponding eigenvector $V$, and $\mu \neq \lambda$ is a real eigenvalue of $A^T$ with corresponding eigenvector $W$, show that$V$ and $W$ are orthogonal: $\langle V, W \rangle = 0$.