Math 313/515, Spring 2005 Jerry L. Kazdan

Homework Set 6, Due Thursday, Feb. 24, 2005
(*Late papers will be accepted until 4 PM on Fri. Feb. 25*)

1. Strang p. 180 #1, 2, 14

2. Strang p. 180 #4, 5

3. Strang p. 181 #10, 11, 17

4. Strang p. 182 #21, 22

5. Strang p. 191 #6

6. Strang p. 192 #10

7. A linear map $R\colon V \to V$ acting on a vector space V is called a *reflection* if $R^2 = I$.

   a) Show that the matrix $R = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ is a reflection. Draw a sketch of $\mathbb{R}^2$ showing the vectors $(1,2)$, $(-1,0)$, (and $(0,3)$ and their images under R. Also indicate both the subspace U of vectors that are map to themselves: $Ru = u$, and the subspace W of vectors that are mapped to their opposites: $Rw = -w$. [From your sketch it is clear that this R is not an *orthogonal reflection* which is when the subspaces U and W are orthogonal.]

   b) Show that the subspaces U and W are orthogonal (so this is an orthogonal projection) if and only if $R = R^*$.

8. [Relation between projections and reflections]. Let P be a projection into a sub-space U and let W be the nullspace of P. Clearly, every vector X and be written as
$$X = PX + (I-P)X, \quad \text{that is,} \quad X = X_1 + X_2,$$
where $X_1 = PX$ and $X_2 = (I-P)X$.

   a) Show that X_1 is in U, that X_2 is in W, and that $P^2X = PX$ for every vector X.

   b) Let R be the related reflection across U, so if X is in U, then $RX = X$, while if X is in W, then $RX = -X$. Show that every vector X can be written as
$$X = X_3 + X_4,$$

where $X_3 \in U$ and $X_4 \in W$. [SUGGESTION: Observe that if we have X_3 and X_4, then you can write RX in terms of X_3 and X_4. Use this to solve for X_3 and X_4 in terms of X and RX].

c) Show that R and P are related by the simple formula $R = 2P - I$. This makes obvious the relation between parts (a) and (b) above.

Problems on Least Squares

9. Strang, p. 215 #1,4

10. Strang, p. 217 #17, 18

11. Strang, p. 229-230 #11, 18, 24

12. Find a plane of the form $z = ax + by + c$ that best fits the following five points: $(0,0,1.1)$, $(1,1,2)$, $(0,1,-0.1)$, $(1,0,3)$, $(0,-1,2.1)$.

13. a) Some experimental data (x_i, y_i) is believed to fit a curve of the form

$$y = \frac{1+x}{a+bx^2}$$

where the parameters a and b are to be determined from the data. The method of least squares does not apply directly to this since the parameters a and b do not appear linearly. Show how to find an equivalent equation to which the method of least squares does apply.

b) Repeat part a) for the *logistic curve* $y = \dfrac{L}{1+e^{a-bx}}$. Here the constant L is assumed to be known. [If $b > 0$, then y converges to L as x increases. Thus the value of L can often be estimated simply by eye-balling a plot of the data for large x.]

14. The comet Tentax, discovered only in 1968, moves within the solar system. The following are observations of its position (r, θ) in a polar coordinate system with center at the sun (here θ is an angle measured in degrees, r in million km):

r	2.70	2.00	1.61	1.20	1.02
θ	48	67	83	108	126

By Kepler's first law the comet should move in a plane orbit whose shape is either an ellipse or hyperbola (this assumes the gravitational influence of the planets is neglected). Thus the polar coordinates (r, θ) satisfy

$$r = \frac{p}{1 - e\cos\theta}$$

where p and e are parameters describing the orbit. Use the data to estimate p and e by the method of least squares. [HINT: Make some (simple) preliminary manipulation so the parameters p and e appear *linearly* so one can then apply the method of least squares.]

Bonus Problem Given a unit vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ in $\mathbb{R}^3$, find an explicit formula for a 3×3 matrix M that rotates a vector $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in $\mathbb{R}^3$, about the "**n** axis" through a specified angle θ. Thus, your matrix M will involve a, b, c, and θ.

[Last revised: February 18, 2005]