Math 313/515, Spring 2005 Jerry L. Kazdan

Homework Set 2, Due Thursday, Jan. 27, 2005

1. Which of the following sets of vectors are bases for $\mathbb{R}^2$?

   a). $\{(0,1),\ (1,1)\}$
   b). $\{(1,0),\ (0,1),\ (1,1)\}$
   c). $\{(1,0),\ (-1,0\}$
   d). $\{(1,1),\ (1,-1)\}$
   e). $\{((1,1),\ (2,2)\}$
   f). $\{(1,2)\}$

2. For which real numbers x do the vectors: $(x,1,1,1)$, $(1,x,1,1)$, $(1,1,x,1)$, $(1,1,1,x)$ *not* form a basis of $\mathbb{R}^4$? For each of the values of x that you find, what is the dimension of the subspace of $\mathbb{R}^4$ that they span?

3. Compute the dimension and find bases for the following linear spaces.

   a) The points $(x,y,z,w) \in \mathbb{R}^4$ that satisfy $y+w=0$
   b) Real symmetric 4×4 matrices.
   c) Cubic polynomials $p(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$ with the property that $p(2) = 0$ and $p(3) = 0$.

4. Find *all* linear maps $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose nullspace is exactly the plane of points (x_1,x_2,x_3) that satisfy $x_1 + 2x_2 - x_3 = 0$.

5. Let C and B be square matrices with C invertible. Show the following.

   a) $(CBC^{-1})^2 = C(B^2)C^{-1}$
   b) Similarly, show that $(CBC^{-1})^k = C(B^k)C^{-1}$ for any $k = 1,2,\dots$.
   c) If B is also invertible, is it true that $(CBC^{-1})^{-2} = C(B^{-2})C^{-1}$? Why?

6. Every real upper triangular $n\times n$ matrix $\left(a_{ij}\right)$ with $a_{ii} = 1$, for $i = 1,2,\dots,n$ is invertible. Proof or counterexample.

7. Let $V \subset \mathbb{R}^{11}$ be a linear subspace of dimension 4 and consider the family $\mathcal{A}$ of all linear maps $L : \mathbb{R}^{11} -> \mathbb{R}^9$ each of whose nullspace contain V.

   Show that $\mathcal{A}$ is a linear space and compute its dimension.

Some Computer Graphics

8. a) Find a 2×2 matrix that rotates the plane by $+45$ degrees (+45 degrees means 45 degrees *counterclockwise*).

   b) Find a 2×2 matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.

   c) Find a 2×2 matrix that reflects across the horizontal axis followed by a rotation the plane by $+45$ degrees.

   d) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.

   e) Find the inverse of each of these maps.

9. a) Find a 3×3 matrix that acts on $\mathbb{R}^3$ as follows: it keeps the x_1 axis fixed but rotates the x_2 x_3 plane by 60 degrees.

   b) Find a 3×3 matrix A mapping $\mathbb{R}^3 \to \mathbb{R}^3$ that rotates the x_1 x_3 plane by 60 degrees and leaves the x_2 axis fixed.

10. Think of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as mapping one plane to another.

    a) If two lines in the first plane are parallel, show that after being mapped by A they are also parallel.

    b) Let Q be the unit square: $0 < x < 1, 0 < y < 1$ and let Q' be its image under this map A. Show that the area$(Q') = ad - bc$. [More generally, the area of any region is magnified by $ad - bc$, which is called the *determinant* of A.]

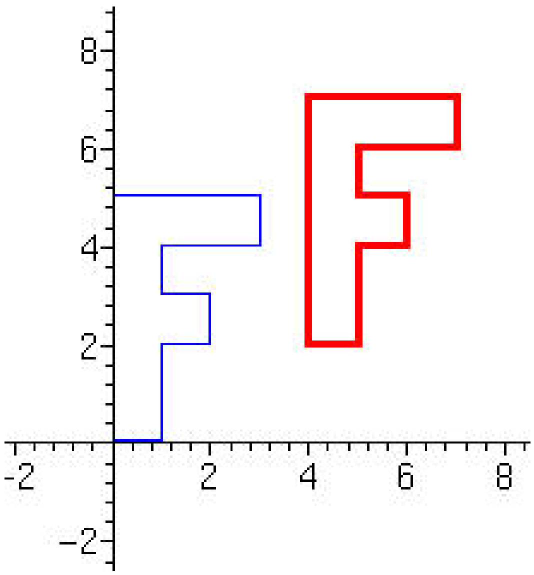
11. Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

$$F(X) = V + AX,$$

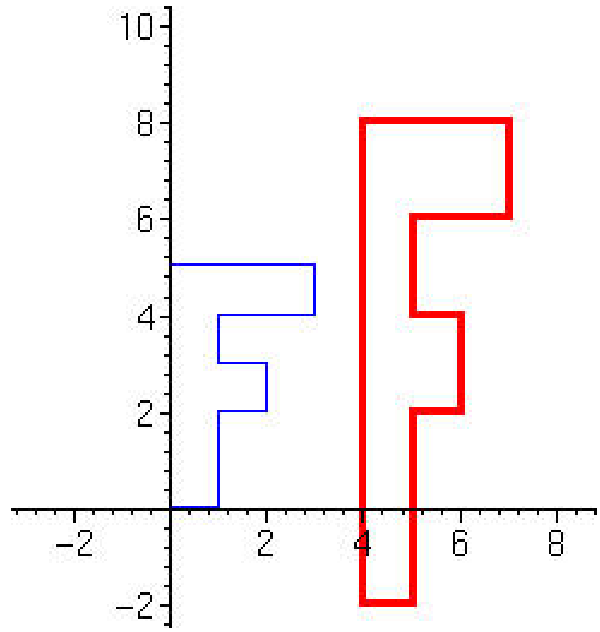
where V is a vector. Note that $F(0) = V$.

Find the vector V and the matrix A that describe each of the following mappings [here the light blue F is mapped to the dark red F]. These pictures were made using Maple (see `http://www.math.upenn.edu/ kazdan/313/display/F-affine.mws`).

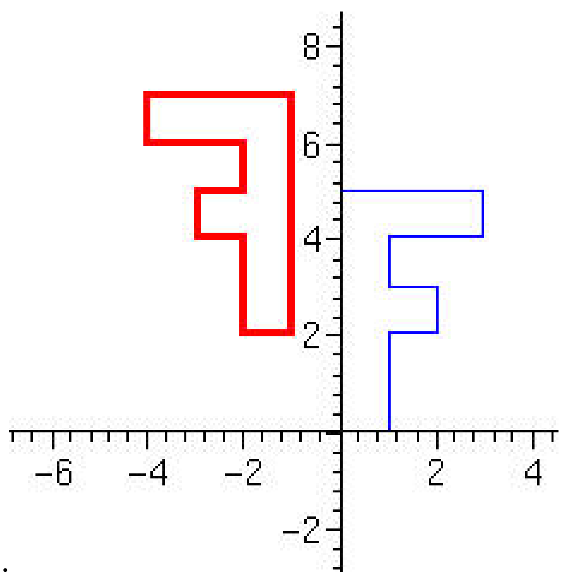
[*continued on next page*]



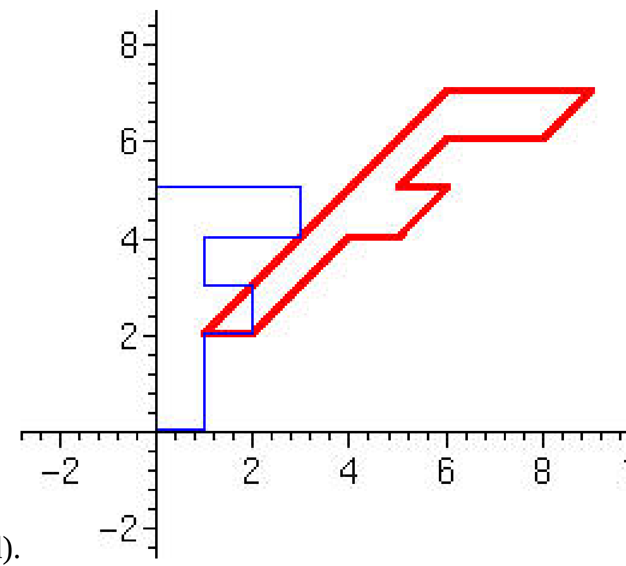
a).



b).



c).



d).