Math 313/515, Spring 2005 Jerry L. Kazdan

Homework Set 11, Due Tuesday, April 19, 2005
(*Late papers will be accepted until 4 PM on Wed. April 20*)

1. [Strang, p. 315 #4] A model for the population of rabbits and wolves. The rabbit population, $r(t)$, increases because of reproduction but decreases because of the wolves. The wolf population, $w(t)$, will increase because of their own reproduction and because there are rabbits to eat. One very simple model might be

$$\frac{dr}{dt} = 6r - 2w, \qquad \frac{dw}{dt} = 2r + w, \quad \text{with, say,} \quad r(0) = w(0) = 30.$$

   a) Find $r(t)$ and $w(t)$.
   b) What is the long-term ration of the number of rabbits to wolves?

2. Let $B = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$. Find a symmetric matrix A so that $A^2 = B$ (thus A is the square root of B).

3. Let B be a real skew-symmetric matrix B (so $B^T = -B$). Show that its eigenvalues are all purely imaginary: $\lambda = ci$, where c is a real number.

4. [Strang, p. 339 #6] Let A be matrix (not necessarily square) whose columns are linearly independent. Show that A^TA is positive definite.

5. If A is a symmetric positive definite matrix and C is any invertible matrix, show that C^TAC is also positive definite.

6. [Strang, p. 327 #5] Find an orthogonal matrix that diagonalizes $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}$

7. Let $Q = x_1^2 + 4x_1x_3 - x_2^2 - 4x_2x_3$. Find an orthogonal matrix R so that in the new coordinates $y = Rx$ this polynomial has the "diagonal" form $Q = ay_1^2 + by_2^2 + cy_3^2$.

8. Let $Q = 3x_1^2 + 8x_1x_2 - 3x_2^2 - x_3^2$. Find an orthogonal matrix R so that in the new coordinates $y = Rx$ this polynomial has the "diagonal" form $Q = ay_1^2 + by_2^2 + cy_3^2$.

9. Let $p(x,y) = 3x^2 - 4xy + 3y^2 - 14x + 16y + 25$. Write this in the form

$$p(x,y) = (\mathbf{X} - \mathbf{X}_0) \cdot A(\mathbf{X} - \mathbf{X}_0) + c,$$

that is, find the axis of symmetry $\mathbf{X}_0$ and the height c. Here $\mathbf{X}$ is the *column* vector $\mathbf{X} = (x, y)$.

10. [Strang, p. 339 #9] Find a 3×3 matrix A so that $\langle X, AX \rangle = 4(x_1 - x_2 + 2x_3)^2$. Also, compute the rank, eigenvalues, and determinant of A.

11. Let A be a positive definite symmetric matrix.
    a) Show that A^2 and A^{-1} are also positive definite.
    b) Show you can find a symmetric matrix C so that $C^2 = A$.

12. If A is any symmetric matrix, show that there is some constant c so that the matrix $A + cI$ is positive definite. Can you find the optimal value of c?

13. If the matrix M in invertible, how are the eigenvalues of M and M^{-1} related? How about the eigenvectors? Prove your assertion.

14. Let $\mathbf{V} = (v_1, \ldots, v_n)$ be a non-zero column vector and let C be the matrix $C = \mathbf{V}\mathbf{V}^T = (v_i v_j)$, so the j^{th} column of C is $v_j \mathbf{V}$.
    a) Show that C is positive semi-definite.
    b) Show that $I + C$ is positive definite and compute its inverse.

15. a) If a (square) matrix A is diagonalizable (that is, it is similar to a diagonal matrix) and if one knows that A is similar to $2A$, show that $A = 0$.
    b) Show that (perhaps to your surprise) the matrices $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ are similar by finding a matrix S so that $S^{-1}AS = B$.

16. [Strang, p. 350 #17] True or False — with a good reason or counterexample.
    a) An invertible matrix can't be similar to a singular matrix.
    b) A symmetric matrix can't be similar to a non-symmetric matrix.
    c) B can't be similar to $-B$ unless $B = 0$.
    d) C can't be similar to $C + I$.

17. Here A and B are both $k \times k$ matrices.

   a) If A is similar to the identity matrix, show that $A = I$.

   b) If A is similar to B, prove that A^2 is similar to B^2.

   c) Give an example of 2×2 matrices where A^2 is similar to B^2 but A is not similar to B.

Bonus Problem 1

For parts a)-b), consider the ellipse $x^2 + y^2/4 = 1$ and the lines $2x + y = c$.

a) Use Matlab or Maple to plot the graphs of the ellipse and the line (on the same plot) for various values of c, both positive and negative.

b) For which value(s) of c does this line intersect the ellipse in exactly one point?

c) Repeat parts a)-b). for the ellipsoid $x^2 + y^2/4 + z^2 = 1$ and, on the same plot, the planes $2x + y + z = c$.

d) More generally, let A be a positive definite symmetric matrix and $\mathbf{b}$ a given non-zero vector. For which value(s) of the constant c does the "plane" $\mathbf{b} \cdot \mathbf{X} = c$ intersect the ellipsoid $\mathbf{X} \cdot A\mathbf{X} = 1$ in exactly one point? [Suggestion: First do the case when $A = I$, then do the case when A is a diagonal matrix. The answer is $c = \pm\sqrt{\mathbf{b} \cdot A^{-1}\mathbf{b}}$.]

Bonus Problem 2 Compute:

$$a).\ \iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+4x^2+9y^2)^2}, \qquad b).\ \iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+5x^2-4xy+5y^2)^2},$$

$$c).\ \iint_{\mathbb{R}^2} e^{-(4x^2+9y^2)}\,dx\,dy, \qquad d).\ \iint_{\mathbb{R}^2} e^{-(5x^2-4xy+5y^2)}\,dx\,dy.$$