Math 313/515, Spring 2005 Jerry L. Kazdan

Homework Set 1, Due Thursday, Jan. 20, 2005

Most of these are routine calisthenics

1. The following system has no solution:

$$\begin{aligned} -x+3y-2z &= 1 \\ -x+4y-3z &= 0 \\ -x+5y-4z &= 0 \end{aligned}$$

   Attempt to solve this system using Gaussian elimination and explain what occurs to indicate the system is impossible to solve.

2. Show how Gaussian elimation leads you to conclude that the following system has infinitely many solutions:

$$\begin{aligned} -x+3y-2z &= 4 \\ -x+4y-3z &= 5 \\ -x+5y-4z &= 6 \end{aligned}$$

3. By solving a 3×3 system, find the coefficients of the parabola $y=a+bx+cx^2$ that passes through the three points $(1,1)$, $(2,2)$, and $(3,0)$.

4. Consider the system of equations

$$\begin{aligned} x+y-\ z &= a \\ x-y+2z &= b. \end{aligned}$$

   a) Find the general solution of the homogeneous equation.
   b) A particular solution of the inhomogeneous equations when $a=1$ and $b=2$ is $x=1$, $y=1$, $z=1$. Find the most general solution of the inhomogeneous equations.
   c) Find some particular solution of the inhomogeneous equations when $a=-1$ and $b=-2$.
   d) Find some particular solution of the inhomogeneous equations when $a=3$ and $b=6$.

   [Remark: After you have done part a), it is possible immediately to write the solutions to the remaining parts.]

5. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

a) Use Gaussian elimination to find the general solution $\mathbf{Z}$ of the homogeneous equation $A\mathbf{Z} = 0$.

b) Find some solution of $A\mathbf{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

c) Find the general solution of the equation in part b).

d) Find some solution of $A\mathbf{X} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ and of $A\mathbf{X} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

e) Find some solution of $A\mathbf{X} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

f) Find some solution of $A\mathbf{X} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. [Note: $\binom{7}{2} = \binom{1}{2} + 2\binom{3}{0}$].

[Remark: After you have done parts a), b) and e), it is possible immediately to write the solutions to the remaining parts.]

6. Let A be a matrix, not necessarily square. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$. Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1$.

b) Find some solution of $A\mathbf{X} = -5\mathbf{Y}_2$.

c) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

d) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X} = 0$.

e) Find *two* solutions of $A\mathbf{X} = \mathbf{Y}_1$.

f) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

g) If A is a square matrix, for any given vector $\mathbf{W}$ can one always find at least one solution of $A\mathbf{X} = \mathbf{W}$? Why?

7. Proof or counterexample. In these L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, so its representation will be as a 2×2 matrix.

a) If L is invertible, then L^{-1} is also invertible.

b) If $LV = 5V$ for all vectors V, then $L^{-1}W = (1/5)W$ for all vectors W.

c) If L is a rotation of the plane by 45 degrees *counterclockwise*, then L^{-1} is a rotation by 45 degrees *clockwise*.

d) If L is a rotation of the plane by 45 degrees counterclockwise, then L^{-1} is a rotation by 315 degrees counterclockwise.

e) The zero map ($0\mathbf{V} = 0$ for all vectors $\mathbf{V}$) is invertible.

f) The identity map ($I\mathbf{V} = \mathbf{V}$ for all vectors $\mathbf{V}$) is invertible.

g) If L is invertible, then $L^{-1}0 = 0$.

h) If $L\mathbf{V} = 0$ for some non-zero vector $\mathbf{V}$, then L is not invertible.

i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: $L = L^{-1}$.

8. Let L be a 2×2 matrix. For each of the following give a proof or counterexample.

   a) If $L^2 = 0$ then $L = 0$.

   b) If $L^2 = L$ then either $L = 0$ or $L = I$.

   c) If $L^2 = I$ then either $L = I$ or $L = -I$.

9. Let L, M, and P be linear maps from the (two dimensional) plane to the plane given in terms of the standard vectors $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$ by:

   $L\mathbf{i} = \mathbf{j}, \quad L\mathbf{j} = -\mathbf{i}$ (rotation by 90 degrees counterclockwise)

   $M\mathbf{i} = -\mathbf{i}, \quad M\mathbf{j} = \mathbf{j}$ (reflection across the vertical axis)

   $NV = -V$ (reflection across the origin)

   a) Draw pictures describing the actions of the maps L, M, and N and the compositions: LM, ML, LN, NL, MN, and NM.

   b) Which pairs of these maps commute?

   c) Which of the following identities are correct—and why?

   1) $L^2 = N$ 2) $N^2 = I$ 3) $L^4 = I$ 4) $L^5 = L$
   5) $M^2 = I$ 6) $M^3 = M$ 7) $MNM = N$ 8) $NMN = L$

   d) Find matrices representing each of the linear maps L, M, and N.

10. Snow White distributed 21 liters of milk among the seven dwarfs. The first dwarf then distributed the contents of his pail evenly to the pails of other six dwarfs. Then the second did the same, and so on. After the seventh dwarf distributed the contents of his pail evenly to the other six dwarfs, it was found that each dwarf had exactly as much milk in his pail as at the start.

    What was the initial distribution of the milk?
    Generalize to N dwarfs.
    [From: K. Splinder *Abstract Algebra with Applications*, Vol. 1, page 192, Dekker (1994)]