

### Some Examples from High School

We first look at some standard examples. While these may seem just like old high school computations, the essential point is not the computations but gaining insight in how to think about these examples as more than sometimes painful computations.

EXAMPLE 0. Diagonal system.

$$\begin{array}{rcl} x & = & -1 \\ 2y & = & 8 \\ 3z & = & 6 \end{array}$$

Although solving these equations is immediate, it is important to understand this simple system as the goal for the “ideal” system of linear equations.

EXAMPLE 1. Upper triangular system.

$$\begin{array}{rcl} x + 2y - z & = & -1 \\ y + 2z & = & 5 \\ 3z & = & 6 \end{array}$$

To solve a special *upper triangular* system of equations like this, first solve the last equation for  $z$ , then the second equation for  $y$  and finally the first equation for  $x$ . One gets  $x = -1$ ,  $y = 1$ ,  $z = 2$ .

It is useful to think of the procedure used to solve this as first modifying this system to the diagonal form of the system in Example 0.

It is also useful to think of solving these equations as the geometry problem of finding the point(s) of intersection of three planes in  $\mathbb{R}^3$ .

EXAMPLE 2. Three equations in three unknowns.

$$\begin{array}{rcl} x + 2y - z & = & -1 \\ 3x + 7y - z & = & 2 \\ x + 3y + 4z & = & 10 \end{array}$$

First use the first equation to eliminate  $x$  from the remaining equation. Thus, add  $-3 \times$  (first equation) to the second equation and  $-1 \times$  (first equation) to the third equation. This gives:

$$\begin{array}{rcl} x + 2y - z & = & -1 \\ y + 2z & = & 5 \\ y + 5z & = & 11 \end{array}$$

Next use the second equation to eliminate from the remaining equation. In this case, add  $-1 \times$  (second equation) to the third equation to find:

$$\begin{aligned}x + 2y - z &= -1 \\y + 2z &= 5 \\3z &= 6\end{aligned}$$

We now have achieved the simple form of Example 1.

This familiar procedure, which is very general, is called **Gaussian elimination**.

EXAMPLE 3. Two equations in three unknowns.

$$x + 2y - z = -1 \quad (1)$$

$$3x + 7y - z = 2 \quad (2)$$

Here we have only used the first two of the three equations from Example 2. Again using Gaussian elimination, use the first equation to eliminate  $x$  from the second equation to get

$$\begin{aligned}x + 2y - z &= -1 \\y + 2z &= 5\end{aligned}$$

Since this is in the simple “upper triangular” form of Example 1, we immediately solve for  $x$  and  $y$  in terms of  $z$ .

$$x = -11 + 5z \quad (3)$$

$$y = 5 - 2z \quad (4)$$

For *every* value of  $z$  we get a solution  $x, y$  of our system. We can think of solving this system of equations (1)–(2) as finding the intersection of the two planes  $x + 2y - z = -1$  and  $3x + 7y - z = 2$ . This intersection is the line defined by (3)–(4).

EXAMPLE 4. Four equations in three unknowns.

$$x + 2y - z = -1$$

$$3x + 7y - z = 2$$

$$x + 3y + 4z = 10$$

$$x + y + z = 0$$

Since there are more equations than unknowns, we suspect we may be in trouble – and we are. We already know the only point that satisfies the first three equations is  $x = -1, y = 1, z = 2$ , but this does not satisfy the last equation, Thus *this system of equations has no solution*.

However if we replaced the last equation by

$$x + y + z = 2$$

(which is satisfied  $x = -1, y = 1, z = 2$ ), then this new system of four equations in three unknowns does have a solution. This is a very rare event.

EXAMPLE 5. Two equations in three unknowns.

$$\begin{aligned}x + 2y - z &= -1 \\2x + 4y - 2z &= 0\end{aligned}$$

Since there are more unknowns than equations, we suspect there will be infinitely many solutions. But we are wrong. The left side of the second equation is twice the first equation, but the right side does not match. Thus *this system of equations has no solution*.

We can think of this problem as asking for the points of intersection of two planes in  $\mathbb{R}^3$  — but these planes are parallel (and do not coincide) so the equations have no solution.

If we replaced the last equation by  $2x - 4y - 2z = -2$  then of course this new system of equations does have a solution.

EXAMPLE 6. Three equations in three unknowns

$$\begin{aligned}x + 2y - z &= -1 \\3x + 7y - z &= 2 \\4x + 9y + 2z &= 0\end{aligned}$$

We anticipate there will be exactly one solution.

Again we are wrong. The left side of the last equation is the sum of the first two equations but the right side is not the sum of the right-hand sides to those equations. Thus *this system of equations has no solution*.

Note that if the right side of the last equation were 1, the equations would have a solution. Since the last equation is then redundant (it is a consequence of the first two equations), this system would even have infinitely many solutions, just as in Example 3.

## Linear Equations as Linear Maps

The next step to gain some insight is to view these linear equations as *Linear Maps*. This will introduce new geometric ingredients that are crucial if we are to pass from tedious computations to real understanding.