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Math 313 — April 29, 2005

Final Exam

Jerry L. Kazdan — 1:30 — 3:30

DIRECTIONS This exam has 12 problems (*10 points each*). Closed book, no calculators – but you may use one 3" × 5" card with notes.

Score	
1	
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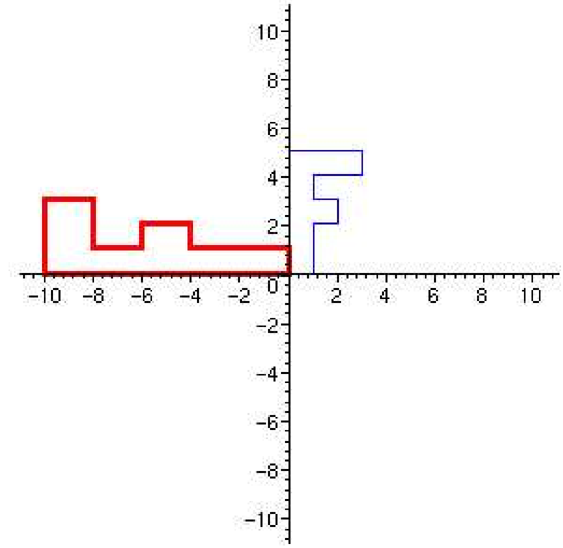
1. Let A be a 3×3 matrix with eigenvalues λ_1, λ_2, λ_3 and corresponding (independent) eigenvectors V_1, V_2, V_3 which we can therefore use as a basis (of course $AV_j = \lambda_j V_j$).
If $X = aV_1 + bV_2 + cV_3$, compute AX, A^2X, and $A^{35}X$ in terms of λ_1, λ_2, λ_3, V_1, V_2, V_3, a, b and c (only).

2. Let $A := \begin{pmatrix} 1 & 4 & 11 & -4 \\ -1 & -2 & -5 & 6 \\ 0 & 4 & 12 & 5 \\ -1 & 2 & 7 & 4 \end{pmatrix}$, $X_0 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. $Y := \begin{pmatrix} -3 \\ 5 \\ 5 \\ 3 \end{pmatrix}$, and $Z := \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}$.
You are given that the vector X_0 is a particular solution of $AX = Y$ and Z is in the nullspace of A.
   a) Find another solution (other than X_0) of $AX = Y$.
   b) If Z is a basis for the nullspace of A, find the general solution of $AX = Y$.

3. Let A be an $n\times n$ matrix of real numbers. Circle each of the following statements that are *NOT* equivalent to: "the matrix A is invertible"? [No justification is needed.]
   a) The columns of A are linearly independent.
   b) The columns of A span $\mathbb{R}^n$.
   c) The only solution of the homogeneous equations $Ax = 0$ is $x = 0$.
   d) The linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ defined by A is 1-1.
   e) The linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ defined by A is onto.
   f) The rank of A is n.
   g) The transpose, A^T, is invertible.

4. Let A, B, and C be $n\times n$ invertible matrices.
   a) Solve the equation $C^{-1}(2I + AM)C = B$ for the matrix M.
   b) If 2 is not an eigenvalue of B, show that M is invertible,

5. a). Find a linear map of the plane, $A : \mathbb{R}^2 -> \mathbb{R}^2$ that does the following transformation of the letter **F** (here the smaller **F** is transformed to the larger one):



b). Find a linear map of the plane that inverts this map, that is, it maps the larger **F** to the smaller.

6. In $\mathbb{R}^3$, compute the distance from the point $(1, 0, 0)$ to the plane $x_1 + 3x_2 - x_3 = 3$.

7. Let $A := \begin{pmatrix} -3 & b \\ b & -3 \end{pmatrix}$, where b is a real constant. To save time, you are given that the eigenvalues of A are $\lambda = -3 \pm b$. Consider the system of differential equations $\dfrac{dU}{dt} = AU$ for the vector $U(t)$. Find *all* values of the parameter b so that $\lim_{t\to\infty} U(t) = 0$.

[Circle the correct answer]
a). All $b > 0$ b). $|b| < 3$ b). $b < 9$ d). $b < 3$ e). $b < -3$ f). $|b| \leq 3$

8. Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

b). If $B = \frac{1}{3}A$, find an invertible matrix P and a diagonal matrix D so that $B = PDP^{-1}$.

c). What can you say about $\lim_{k\to\infty} B^k$? (Please briefly justify your assertion.)

9. A multinational company has branches in the US., Japan, and Europe. In 1990, it had assets of \$5 million: \$3 million are in the U.S. and \$2 million in Europe. Each year 1/2 the U.S. money stays home, 1/4 goes to both Japan and Europe. For Japan and Europe, 1/2 stays home and 1/2 is sent to the U.S.

   a). Find the transition matrix of this Markov chain.

   b). Find the limiting distribution of the \$5 million as the world ends.

10. Say you seek a parabola with the special form $y = a(x-1)^2 + b$ to pass through the three data points $(0,\,2)$, $(1,\,0)$, $(2,\,3)$.

    a) Write the (over-determined) system of equations you would like to solve ideally.

    b) Using the method of least squares write the *normal* equations for the coefficients a, b.

    c) Explicitly find the coefficients a and b.

11. Let A be a matrix (not necessarily square) whose columns are linearly independent. Show that the matrix $A^T A$ is positive definite.

12. Let A be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$.

    a) Show that $\dfrac{\langle x,\, Ax\rangle}{\|x\|^2} \ge \lambda_1$.

    b) Let $B = A - cI$. If $c < \lambda_1$ show that B is positive definite.