Problem Set 9

Due: In class Thursday, Thurs. April 10 *Late papers will be accepted until 1:00 PM Friday.*

For the coming week, please re-read Chapter 7 and Chapter 8.1-8.2.

1. Let $z = x + iy$ be a complex number. For which real numbers x, y is $|e^z| < 1$?

2. This asks you to come up with four examples. In each case, find a real matrix (perhaps 2×2) that is:
   a) Both invertible and diagonalizable.
   b) Not invertible, but diagonalizable.
   c) Not diagonalizable but is invertible.
   d) Neither invertible nor diagonalizable.

3. Let A and B be $n \times n$ real positive definite matrices and let $C := tA + (1-t)B$. If $0 \le t \le 1$, show that C is also positive definite. [This is simple. No "theorems" are needed.]

4. Let A be an $m \times n$ matrix, and suppose $\vec{v}$ and $\vec{w}$ are orthogonal eigenvectors of $A^T A$. Show that $A\vec{v}$ and $A\vec{w}$ are orthogonal.

5. [Bretscher, Sec. 7.3 #28] Let $B := \begin{pmatrix} k & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & k \end{pmatrix}$ where k is an arbitrary constant. Find the eigenvalue(s) of B and determine both their algebraic and geometric multiplicities. [Note: First try the analogous 2×2 case.]

6. [Bretscher, Sec. 8.1 #24] Find an orthonormal eigenbasis for $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

7. [Bretscher, Sec. 8.1 #38] Let A be a symmetric 2×2 matrix with eigenvalues -2 and 3 and $u \in \mathbb{R}^2$ any unit vector. What are the possible values of $\langle u, Au \rangle$? Illustrate your answer in terms of the unit circle and its image under A.

8. [Bretscher, Sec. 7.6 #18] If $\vec{x}(t+1) = A\vec{x}(t)$, where $A := \begin{pmatrix} -0.8 & 0.6 \\ -0/8 & -0.8 \end{pmatrix}$ and $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, find a real closed formula for the trajectory $\vec{x}(t)$. Also, draw a rough sketch.

9. If $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, solve $\dfrac{d\vec{x}}{dt} = A\vec{x}$ with initial condition $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

10. [Bretscher, Sec. 7.5 #24] Find all the eigenvalues of $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{pmatrix}$.

11. If a real matrix A can be diagonalized by an orthogonal matrix, show A is symmetric.

12. Of the following four matrices, which can be orthogonally diagonalized; which can be diagonalized (but not orthogonally); and which cannot be diagonalized at all. Identify these – *fully explaining your reasoning.*

$$A = \begin{pmatrix} 0 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

13. [Bretscher, Sec. 8.2 #26] Consider the quadratic polynomial $Q(\vec{x}) := \langle \vec{x}, A\vec{x} \rangle$, where A is a real $n \times n$ symmetric matrix. "If for some vector $\vec{v} \neq 0$ we know that $Q(\vec{v}) = 0$, then A cannot be invertible." Proof or counterexample.

14. [Bretscher, Sec. 7.5 #14] Let $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$. Find an invertible matrix S so that $S^{-1}AS = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

15. Let A be an $n \times n$ matrix that commutes with *all* $n \times n$ matrices, so $AB = BA$ for all matrices B. Show that $A = cI$ for some scalar c. [Suggestion: Let $\vec{v}$ be an eigenvector of A with eigenvalue λ and use Homework Set 2 #16c].

Bonus Problems

[Please give this directly to Professor Kazdan]

B-1 Let A be an $n \times n$ matrix all of whose elements are 1 and let $L := I + A$.

a) Why is L invertible?

b) Find and explicit formula for L^{-1}. [SUGGESTION: Let $\vec{v}$ be a column vector of all 1's and note that $\vec{v}$ is a basis for the image of A. Thus $A\vec{x} = c\vec{v}$ for some scalar c that depends on $\vec{x}$. But if $L\vec{x} = \vec{y}$, then $\vec{x} = \vec{y} - A\vec{x} = \vec{y} - c\vec{v}$ so all you need to do is find the scalar c.]

[Last revised: May 4, 2014]