Math 312, Spring 2014 Jerry L. Kazdan

Problem Set 4

DUE: In class Thursday, Thurs. Feb. 13. *Late papers will be accepted until 1:00 PM Friday.*

Reminder: Exam 1 is on Tuesday, Feb. 18, 9:00–10:20. No books or calculators but you may always use one 3" × 5" card with handwritten notes on both sides.

For the coming week, please review Chapter 4 Sections 4.1 and 4.2. We have already covered most of Chapter 4.
Later we will return in greater detail to the material in Section 3.4 on Coordinates.

1. Find a basis for the linear space of matrices $\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$ with the property that $a+d=0$. What is the dimension of this space.

2. Find a linear map $L:\mathbb{R}^3\to\mathbb{R}^3$ whose kernel is exactly the plane

$$\{(x_1,\, x_2,\, x_3)\subset\mathbb{R}^3 \mid x_1+2x_2-x_3=0\}.$$

3. LIKE BRETSCHER, SEC. 4.2 #66 Find the kernel of the map $T: C^\infty(\mathbb{R})\to C^\infty(\mathbb{R})$ defined by $T(u):=u'-4u$. What is the dimension of the kernel?
   a) Repeat this for $Tu:=u''-4u$.

4. We want to approximately compute $\int_0^2 \frac{1}{1+x^2}\,dx$ by partitioning the interval $0\le x\le 2$ into four sub-intervals whose end point are $x_0=0, x_1=0.5$, $x_2=1$, $x_3=1.5$, $x_4=2$. of width $h=x_{i+1}-x_i=1/2$. In each sub-interval replace the integrand by a simpler function.

   TRAPEZOIDAL RULE: Approximate the function $f(x)$ in each sub-interval $[x_i,x_{i+1}]$ by a straigh line joining its end points: $(x_1,\, f(x_i))$, $(x_{i+1},\, f(x_{i+1}))$.

   SIMPSON'S RULE This works with two sub-intervals at a time, say $x_0\le x\le x_1$ and $x_1\le x\le x_2$ and uses a parabola,

$$p(x):=a+bx+cx^2$$

   that passes through the three points $(x_0,\, y_0)$, $(x_1,\, y_1)$, and $(x_2,\, y_2)$. The idea is to approximate the area under the function in the interval $x_0\le x\le x_2$ by the area under the parabola.

5. Find a basis for the space $\mathcal{P}_4$ of polynomials $p(x)$ degree at most 4 with the properties $p(1)=0$ and $p(3)=0$. What is the dimension of this space?

6. In class we considered the interpolation problem of finding a polynomial of degree n passing through $n+1$ specified distinct points in the plane. To be definite, take $n=3$, and say our points are (a_1,b_1), (a_2,b_2), (a_3,b_3), and (a_4,b_4). This problem involves $\mathcal{P}_3$, and so we could work in the usual basis $\{1,x,x^2,x^3\}$. However, it is easier to use the *Lagrange basis*. The point of this problem is to see vividly why choosing a basis adapted to the problem may involve much less work.

   a) Setup the linear equations you would need to solve to find the polynomial of degree 3 passing through the points $(0,-3)$, $(1,-1)$, $(2,11)$, and $(-1,-7)$ if you use the usual basis $\{1,x,x^2,x^3\}$. But don't take time to solve these.

   b) Solve the same problem explicitly using the Lagrange basis.

7. [[Bretscher, Sec. 4.2 #70] Does there exist a polynomial $f(t)$ of degree at most 4 such that $f(2)=3$, $f(3)=5$, $f(5)=7$, $f(7)=11$, and $f(11)=2$? If so, how many such polynomials are there? [: Note: This problem only asks if such a polynomial exists. It is not asking you to find it.]

8. Let $\mathcal{P}_2$ be the linear space of polynomials of degree at most 2 and $T:\mathcal{P}_2\to\mathcal{P}_2$ be the transformation

$$(T(p))(t)=\frac{1}{t}\int_0^t p(s)\,ds.$$

   For instance, if $p(t)=2+3t^2$, then $T(p)=2+t^2$.

   a) Prove that T is a linear transformation.

   b) Find the kernel of T, and find its dimension.

   c) Find the range (=image) of T, and compute its dimension.

   d) Verify the dimension of the kernel and the dimension of the range add up to what you would expect.

   e) Using the standard basis $\{1,\,t,\,t^2\}$ for $\mathcal{P}_2$, represent the linear transformation T as a matrix A.

   f) Using your matrix represention from (e), find $T(p)$ where $p(t)=t-2$.

The remaining problems are from the Lecture notes on Vectors

`http://www.math.upenn.edu/~kazdan/312S13/notes/vectors/vectors10.pdf`

9. [p. 8 #5] The origin and the vectors X, Y, and $X+Y$ define a parallelogram whose diagonals have length $X+Y$ and $X-Y$. Prove the *parallelogram law*

$$\|X+Y\|^2+\|X-Y\|^2=2\|X\|^2+2\|Y\|^2;$$

   This states that in a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the four sides.

10. [p. 8 #6] (Math 240 Review)

   a) Find the distance from the straight line $3x - 4y = 10$ to the origin. [It may help to observe that this line is parallel to the plane $3x - 4y = 0$, whose normal vector is clearly $\vec{N} = (3,\, -4)$.]

   b) Find the distance from the plane $ax + by + cz = d$ to the origin (assume the vector $\vec{N} = (a, b, c) \neq 0$).

11. [p. 8 #8]

   a) If X and Y are real vectors, show that

   $$\langle X,\, Y\rangle = \frac{1}{4}\left(\|X+Y\|^2 - \|X-Y\|^2\right).$$

   This formula is the simplest way to recover properties of the inner product from the norm.

   b) As an application, show that if a square matrix R has the property that it preserves length, so $\|RX\| = \|X\|$ for every vector X, then it preserves the inner product, that is, $\langle RX,\, RY\rangle = \langle X,\, Y\rangle$ for all vectors X and Y.

12. [p. 9 #10] (Also done in class)

   a) If a certain matrix C satisfies $\langle X,\, CY\rangle = 0$ for *all* vectors X and Y, show that $C = 0$.

   b) If the matrices A and B satisfy $\langle X,\, AY\rangle = \langle X,\, BY\rangle$ for all vectors X and Y, show that $A = B$.

13. [p. 9 #11–12] A matrix A is called *anti-symmetric* (or skew-symmetric) if $A^* = -A$.

   a) Give an example of a 3×3 anti-symmetric matrix (other than the trivial $A = 0$).

   b) If A is any anti-symmetric matrix, show that $\langle X,\, AX\rangle = 0$ for all vectors X.

[Last revised: February 21, 2014]