Homework 3 Solutions

1. Let A, B, and C be $n \times n$ matrices with A and C invertible. Solve the equation $ABC = I - A$ for B.

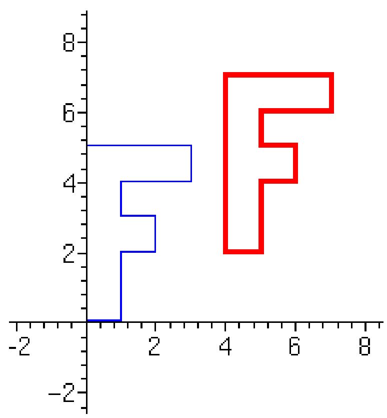
   SOLUTION: $B = A^{-1}(I - A)C^{-1}$. You can rewrite this in various ways – but I won't. However, one must be careful since the matrices A, B, and C are not assumed to commute.

2. If a square matrix M has the property that $M^4 - M^2 + 2M - I = 0$, show that M is invertible. [Suggestion: Find a matrix N so that $MN = NM = I$. This is very short.]

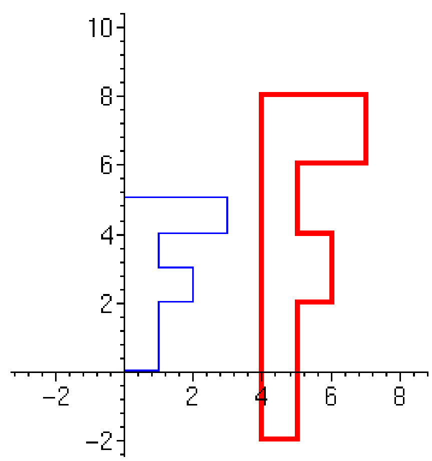
   SOLUTION The given equation implies that $M(M^3 - M + 2I) = I$ hence for $N = M^3 - M + 2I$ we have $MN = NM = I$, hence M is invertible with inverse N.

3. Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,
$$F(X) = V + AX,$$
   where V is a vector. Note that $F(0) = V$.

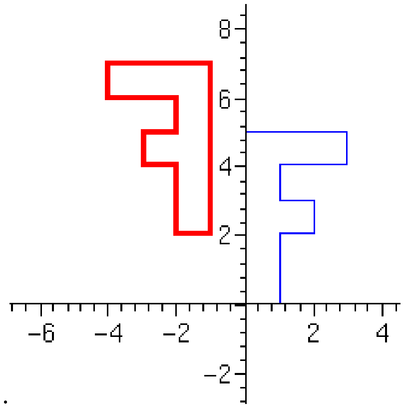
   Find the vector V and the matrix A that describe each of the following mappings [here the light blue F is mapped to the dark red F].



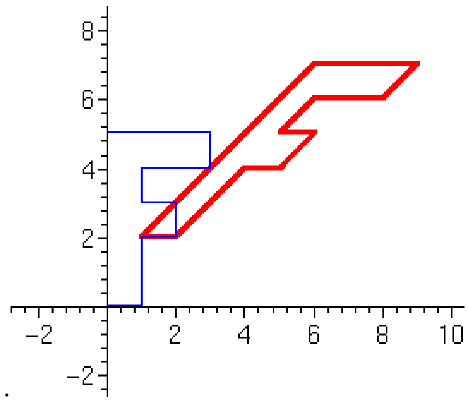
a).



b).



c).



d).

SOLUTION:

a). $V = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $A = I$ b). $V = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

c). $V = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ d). $V = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

4. Use Theorems from Section 3.3 (or from class) to explain the following carefully.
   a) If V and W are subspaces with V contained inside W, why is $\dim V \le \dim W$?
   b) If $\dim V = \dim W$, explain why $V = W$.

   SOLUTION

   a) Let $\dim V = m$, $\dim W = n$. Now, if $\mathcal{B}$ is a basis for V then $\mathcal{B}$ will also be a subset of linearly independent vectors of W. Also, we know that for every linearly independent subset of W the number of its elements can be at most equal to the dimension of W, i.e. $m \le n$, from the definitions of the dimension and basis of a vector space.
   b) If $\dim V = \dim W$ and $\mathcal{B}$ is a basis for V then $\mathcal{B}$ spans V. Since V is a subspace of W, it means we can extend $\mathcal{B}$ to be a basis of W, but by adding any vector we obtain a linear dependent set since $\dim W = \dim V = \#\mathcal{B}$, so we $\mathcal{B}$ must span W as well. Hence $V = \operatorname{span}\{\mathcal{B}\} = W$.

5. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$, so $BA : \mathbb{R}^3 \to \mathbb{R}^3$ and $AB : \mathbb{R}^2 \to \mathbb{R}^2$.
   a) Why must there be a non-zero vector $\vec{x} \in \mathbb{R}$ such that $A\vec{x} = 0$.
   b) Show that the 3×3 matrix BA can not be invertible.
   c) Give an example showing that the 2×2 matrix AB might be invertible.

   SOLUTION

   a) Since $3 = \dim \mathbb{R}^3 = \dim(\ker A) + \dim(\operatorname{im} A)$ and $\dim(\operatorname{im} A) \le 2$, then $\dim(\ker A) \ge 1$.

b) If $\vec{x} \in \ker A$ then since B linear map we get that $\vec{x} \in \ker BA$ so from (a) we obtain that $\ker BA$ is not trivial, hence BA not invertible.

c) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then AB as a linear map is the identity hence it's invertible while for BA easily we can verify that is not invertible.

6. Let A be a square matrix. If A^2 is invertible, show that A is invertible. [Note: You cannot use the formula $(AB)^{-1} = B^{-1}A^{-1}$ because it presumes you already know that both A and B are invertible. For non-square matrices, it is possible for AB to be invertible while neither A nor B are (see the last part of the previous problem).]

   SOLUTION [METHOD 1] Since A^2 is invertible, there exists a square matrix B such that $A^2B = I$ hence $A(AB) = I$. Similarly, $(BA)A = I$. Thus A is invertible with inverse AB.

   [METHOD 2] $\ker A^2 = 0$ so $\ker A = 0$. Since A is a square matrix, then it is invertible.

   [METHOD 3] For any y there is a solution x of $A^2x = y$. Thus $w := Ax$ is a solution of $Aw = y$ so A is onto. Since A is a square matrix then it is invertible.

7. [BRETSCHER, SEC. 2.4 #35] An $n \times n$ matrix A is called *upper triangular* if all the elements below the *main diagonal*, a_{11}, a_{22}, $\ldots a_{nn}$ are zero, that is, if $i > j$ then $a_{ij} = 0$.

   a) Let A be the upper triangular matrix

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

   For which values of a, b, c, d, e, f is A invertible?

   SOLUTION: As always, in thinking about the invertability I think of solving the equations $A\vec{x} = \vec{y}$. In this case, the equations are

$$\begin{aligned} ax_1 + bx_2 + cx_3 &= y_1 \\ + dx_2 + ex_3 &= y_2 \\ fx_3 &= y_3 \end{aligned}$$

   Clearly, to always be able to solve the last equation for x_3 we need $f \neq 0$. This gives us x_3, which we use in the second equation. It then can always be solved for x_2 if (and only if) $d \neq 0$. Inserting the values of x_2 and x_3 in the first equation, it can always be solved for x_1 if (and only if) $a \neq 0$.
   Summary: An upper triangular matrix A is invertible if and only if none of its diagonal elements are 0.

b) If A is invertible, is its inverse also upper triangular?

SOLUTION: In the above computation, notice that x_3 only depends on y_3. Then x_2 only depends on y_2 and y_3. Finally, x_1 depends on y_1, y_2, and y_3. Thus the inverse matrix is also upper triangular.

c) Show that the product of two $n \times n$ upper triangular matrices is also upper triangular.

SOLUTION: Try the 3×3 case.

The general case is the same – but takes some thought to write-out clearly and briefly. It is a consequence of three observations:

1. A matrix $C := \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$ is upper-triangular if all the elements below the main diagonal are zero, that is, $c_{jk} = 0$ for all $j > k$.

2. For any matrices, to compute the product AB, the jk element is the dot product of the j^{th} row of A with the k^{th} column of B.

3. For upper-triangular matrices:

the j^{th} row of A is $(0, \ldots 0, a_{jj}, \ldots, a_{jn})$ while the k^{th} column of B is $\begin{pmatrix} b_{1k} \\ \vdots \\ b_{kk} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

For $j > k$, take the dot product of these vectors. The result is now obvious.

d) Show that an upper triangular $n \times n$ matrix is invertible if none of the elements on the main diagonal are zero.

SOLUTION: This is the same as part a). The equations $A\vec{x} = \vec{y}$ are

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1\,n-1}x_{n-1} + a_{1n}x_n &= y_1 \\ a_{22}x_2 + \cdots + a_{2\,n-1}x_{n-1} + a_{2n}x_n &= y_2 \\ \vdots \qquad \vdots \qquad & \quad \vdots \\ a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n &= y_{n-1} \\ a_{nn}x_n &= y_n. \end{aligned}$$

To begin, solve the last equation for x_n. This can always be done if (and only if) $a_{nn} \neq 0$. Then solve the second from the last for x_{n-1}, etc. This computation also proves the converse (below).

As in part b), the inverse, if it exists, is also upper triangular.

e) Conversely, if an upper triangular matrix is invertible show that none of the elements on the main diagonal can be zero.

SOLUTION: This follows from the reasoning of the previous part. Say $a_{jj} = 0$ but none of the diagonal elements for larger j are zero. Then as in the previous part, we can solve for x_n, then $x_{n-1}, \ldots, x_{j+1}$ in terms of $y_n, \ldots, y_{j+1}$. But since $a_{jj} = 0$, the j^{th} equation

$$0x_j + a_{(j+1)(j+1)}x_{j+1} + \cdots + a_{nn}x_n = y_j$$

can only be solved if y_j satisfies the above condition, so A cannot be invertible.

ALTERNATE Using determinants (which we have not yet covered), briefly we can verify this since for an upper triangular matrix the determinant is equal to the product of the elements on the main diagonal.

8. [SEE BRETSCHER, SEC. 3.2 #6] Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and let $W = U \cap V$. Find all possible values for the dimension of W.

SOLUTION: Let $e_1 = (1,0,0,0,0)$, $e_2 = (0,1,0,0,0)$,..., $e_5 = (0,0,0,0,1)$ be the standard basis for $\mathbb{R}^5$ and say U is spanned by e_1 and e_2.

If V is also spanned by e_1 and e_2 the dimension of W is 2, clearly the largest possible.

If V is spanned by e_1 and e_3 the dimension of W is 1.

If V is spanned by e_3 and e_4 the dimension of W is 0. They intersect only at the origin.

9. [SEE BRETSCHER, SEC. 3.2 #50] Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and define the set $W := U + V$ as the set of all vectors $w = u + v$ where $u \in U$ and $v \in V$ can be any vectors.

a) Show that W is a linear space.

SOLUTION: Since the sum of two vectors in U is in U and the sum of two vectors in V is also in V, then the sum of two vectors in W is also in W
Similarly, if $\vec{w} = \vec{u} + \vec{v} \in W$, then so is $c\vec{w} = c\vec{u} + c\vec{v}$ for any scalar c.

b) Find all possible values for the dimension of W.

SOLUTION: We use the notation of the previous problem.
If V is also spanned by e_1 and e_2 the dimension of W is 2, clearly the smallest possible.
If V is spanned by e_1 and e_3 the dimension of W is 3.
If V is spanned by e_3 and e_4 the dimension of W is 4. This is the largest possible.

10. Say you have k linear algebraic equations in n variables; in matrix form we write $A\vec{x} = \vec{y}$. Give a proof or counterexample for each of the following.

a) If $n = k$ (same number of equations as unknowns), there is always *at most one* solution.

SOLUTION: False. $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ are both counterexamples. It is true only if A is invertible.

b) If $n > k$ (more unknowns than equations), you can *always* solve $A\vec{x} = \vec{y}$.

SOLUTION: False. Counterexamples: $A = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{smallmatrix}\right)$.

c) If $n > k$ (more unknowns than equations), the nullspace of A has dimension greater than zero.

SOLUTION: True. For $A\vec{x} = \vec{y}$, if there are more unknowns than equations, then the homogeneous equation $A\vec{x} = 0$ always has a a solution other than the trivial solution $\vec{x} = 0$.

d) If $n < k$ (more equations than unknowns), then for *some* $\vec{y}$ there is *no* solution of $A\vec{x} = \vec{y}$.

SOLUTION: True. If $A : \mathbb{R}^n \to \mathbb{R}^k$, then the dimension of the image of A is at most n. Thus, if $n < k$ then A cannot be onto.

e) If $n < k$ (more equations than unknowns), the *only* solution of $A\vec{x} = 0$ is $\vec{x} = 0$.

SOLUTION: False. Counterexamples: $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{smallmatrix}\right)$.

11. [BRETSCHER, SEC. 3.3 #30] Find a basis for the subspace of $\mathbb{R}^4$ defined by the equation $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

SOLUTION: Solve this for, say, $x_2 = 2x_1 + 2x_3 + 4x_4$. Then a vector $\vec{x}$ is in the subspace if (and only if) for any choice of x_1, x_3, and x_4

$$\vec{x} = \begin{pmatrix} x_1 \\ 2x_1 + 2x_3 + 4x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 0 \\ 4 \\ 0 \\ 1 \end{pmatrix} x_4.$$

The three column vectors on the right are a basis for this subspace: dimension is 3.