Math 312, Spring 2014 Jerry L. Kazdan

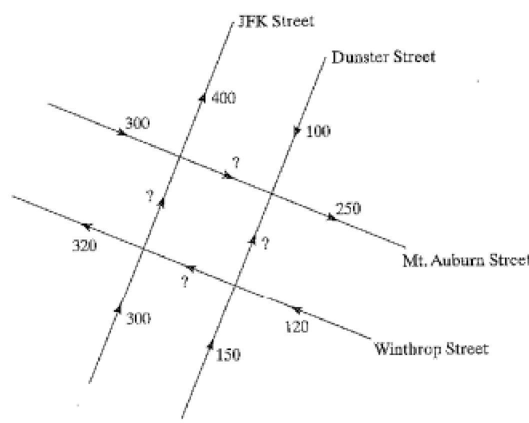
Problem Set 2

DUE: In class Thursday, Jan. 30. *Late papers will be accepted until 1:00 PM Friday.*

Lots of problems. Fortunately, most are really short.

For the coming week, please read Sections 3.1, 3.2, and 3.3 in the text. Much of this should be a repeat from Math 240.
Just as last week, in addition to the problems below, you should also know how to solve *all* of the problems at the beginning of each of the problem sets for each of the Section 3.1-3.3. Most are simple mental exercises. Later we will return in more detail to orthogonal projections, reflections, and rotations.
Note that I will **not** use *reduced row-echelon form* in this course. Although it is splendid for computational efficiency, I find that it short-circuits understanding because it focuses on computational details rather than broader concepts.

1. [Bretscher, Sec. 1.2 #44] The sketch represents a maze of one-way streets in a city. The traffic volume through certain blocks during an hour has been measured. Suppose that the number of vehicles leaving this area during this hour was exactly the same as the number of vehicles entering it.
   What can you say about the traffic volume at the four locations indicated by question marks? Can you determine exactly how much traffic there was on each block? If not, find the highest and lowest possible traffic volumes.



2. Consider the system of equations

$$\begin{aligned} x + y - \ z &= a \\ x - y + 2z &= b \\ 3x + y \qquad &= c \end{aligned}$$

   a) Find the general solution of the homogeneous equation.
   b) If $a = 1$, $b = 2$, and $c = 4$, then a particular solution of the inhomogeneous equations is $x = 1$, $y = 1$, $z = 1$. Find the most general solution of these inhomogeneous equations.
   c) If $a = 1$, $b = 2$, and $c = 3$, show these equations have *no* solution.
   d) If you view these equations as defining a linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$, find a basis for $\ker(A)$ and for $\operatorname{image}(A)$.

3. Let A and B both be $n \times n$ matrices. What's wrong with the formula $(A+B)^2 = A^2 + 2AB + B^2$? Prove that if this formula is valid for A and B, then A and B commute.

4. [Bretscher, Sec.2.2 #17] Let $A := \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, where $a^2 + b^2 = 1$. Find two perpendicular non-zero vectors $\vec{v}$ and $\vec{w}$ so that $A\vec{v} = \vec{v}$ and $A\vec{w} = -\vec{w}$ (write the entries of $\vec{v}$ and $\vec{w}$ in terms of a and b). Conclude that thinking of A as a linear map it is an orthogonal reflection across the line $\mathcal{L}$ spanned by $\vec{v}$.

5. [Bretscher, Sec.2.2 #31] Find a nonzero 3×3 matrix A so that $A\vec{x}$ is perpendicular to $\vec{v} := (1, 2, 3)$ for all vectors $\vec{x} \in \mathbb{R}^3$. [There are many examples. You are only asked to find one example.]

6. [Bretscher, Sec.2.3 #48]
   a) If $A := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $B := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, compute AB and A^{10}.
   b) Find a 2×2 matrix A so that $A^{10} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

7. Which of the following subsets of $\mathbb{R}^2$ are actually linear subspaces? Explain.
   a) $\{(x, y) \mid xy = 0\}$
   b) $\{(x, y) \mid x \text{ and } y \text{ are both integers}\}$
   c) $\{(x, y) \mid x + y = 0\}$
   d) $\{(x, y) \mid x + y = 2\}$
   e) $\{(x, y) \mid x + y \geq 0\}$

8. Which of the following sets are linear spaces? Why?
   a) $\{\vec{x} = (x_1, x_2, x_3) \text{ in } \mathbb{R}^3 \text{ with the property } x_1 - 2x_3 = 0\}$
   b) The set of solutions x of $Ax = 0$, where A is an $m \times n$ matrix.
   c) The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)\, dx = 0$.
   d) The set of solutions $y = y(t)$ of $y'' + 4y' + y = 0$ (you are *not* being asked to actually find these solutions).
   e) The set of all 2×3 matrices with real coefficients?
   f) The set of all 2×2 invertible real matrices?

9. Proof or counterexample. In these L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, so its representation will be as a 2×2 matrix.
   a) If L is invertible, then L^{-1} is also invertible.

b) If $L\vec{v} = 5\vec{v}$ for all vectors $\vec{v}$ and $M\vec{w} = (1/5)\vec{w}$ for all vectors $\vec{w}$, then L is invertible and $L^{-1} = M$.

c) If L is a rotation of the plane by 45 degrees *counterclockwise*, then L^{-1} is a rotation by 45 degrees *clockwise*.

d) If L is a rotation of the plane by 45 degrees counterclockwise, then L^{-1} is a rotation by 315 degrees counterclockwise.

e) The zero map ($0\vec{v} := 0$ for all vectors $\vec{v}$) is invertible.

f) The identity map ($I\vec{v} := \vec{v}$ for all vectors $\vec{v}$) is invertible.

g) If L is invertible, then $L^{-1}0 = 0$.

h) If $L\vec{v} = 0$ for some non-zero vector $\vec{v}$, then L is not invertible.

i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: $L = L^{-1}$.

10. Let V and W be linear spaces and $T : V \to W$ a linear map.

a) Assume the kernel of T is trivial, that is, the only solution of the homogeneous equation $T\vec{x} = 0$ is $\vec{x} = 0$. Prove that if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$.

b) Conversely, if T has the property that "if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$," show that the kernel of T is trivial.

11. Say $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent vectors in $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map.

a) Show by an example, say for $n = 2$, that $T\vec{v}_1, \ldots, T\vec{v}_n$ need not be linearly independent.

b) However, show that if the kernel of T is trivial, then these vectors $T\vec{v}_1, \ldots, T\vec{v}_n$ are linearly independent.

12. [Like Bretscher, Sec. 2.4 #40]. Let A be a matrix, not necessarily square.

a) If A has two equal rows, show that it is not onto (and hence not invertible).

b) If A has two equal columns, show that it is not one-to-one (and hence not invertible).

13. Let V be the linear space of smooth real-valued functions $u(x)$ and $L : V \to V$ the linear map defined by $Lu := u'' + u$.

a) Compute $L(e^{2x})$ and $L(x)$.

b) Find particular solutions of the inhomogeneous equations

$$a).\ u'' + u = 7e^{2x}, \qquad b).\ w'' + w = 4x, \qquad c).\ z'' + z = 7e^{2x} - 3x$$

14. Let $A : \mathbb{R}^3 \to \mathbb{R}^5$ and $B : \mathbb{R}^5 \to \mathbb{R}^2$.

   a) What are the maximum and minimum values for the dimension of the kernels of A, B, and BA?

   b) What are the maximum and minimum values for the dimension of the images of A, B, and BA?

15. Think of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as mapping one plane to another. If two lines in the first plane are parallel, show that after being mapped by A they are also parallel – although they might coincide.

   [REMARK: The simplest way to describe a straight line in $\mathbb{R}^2$ (or even $\mathbb{R}^n$) that passes through two points, say P and Q, is to think of this line as the path $\vec{x}(t)$ of a particle at time t with $\vec{x}(t) = P + t\vec{v}$ where $\vec{v} = Q - P$ and $-\infty < t < \infty$. For any point $\hat{P}$ the line $\vec{y}(t) = \hat{P} + t\vec{w}$ is *parallel* to $\vec{x}(t)$ if $\vec{w} = c\vec{v}$ for some scalar c. Here we are assuming $\vec{v} \neq 0$ and $\vec{w} \neq 0$.]

16. In $\mathbb{R}^n$ let $\vec{e}_1 = (1, 0, 0, \ldots, 0)$, $\vec{e}_2 = (0, 1, 0, \ldots, 0)$ and let $\vec{v}$ and $\vec{w}$ be any non-zero vectors in $\mathbb{R}^n$.

   a) Find an invertible matrix A with $A\vec{e}_1 = \vec{e}_2$ (there are many such matrices).

   b) Show there is an invertible matrix B with $B\vec{e}_1 = \vec{v}$ (there are many such matrices). [If $\vec{v} = (v_1, v_2, \ldots, v_n)$, for simplicity you may assume that $v_1 \neq 0$.]

   c) Show there is an invertible matrix M with $M\vec{w} = \vec{v}$ (there are many such matrices).

[Last revised: January 29, 2014]