Math 312, Spring 2014 Jerry L. Kazdan

Problem Set 11

DUE: In class Thursday, Thurs. April 24. *Late papers will be accepted until 1:00 PM Friday.*

For the coming week, please read Chapter 8.3 in the text and the notes:
`https://canvas.upenn.edu/courses/1234214/files/46904271/download?wrap=1`
containing the Linear Programming sections in Strang's *Linear Algebra* text.

1. If A is a positive definite matrix, show that A^k is also positive definite for any integer k (including negative integers). [Note that a positive definite matrix is *always* assumed to be symmetric.]

2. If A is a positive definite matrix, show that for any invertible matrix C, the matrix C^*AC is also positive definite.

3. For a linear programming problem, say the feasible set has the constraints

$$x + 2y \geq 6, \qquad 2x + y \geq 6, \qquad x \geq 0, \qquad y \geq 0.$$

   Sketch this. What are the three "corners" of this set?

4. In a linear programming problem, show that the feasible set constrained by

$$2x + 5y \leq 3, \qquad -3x + 8y \leq -5, \qquad x \geq 0, \qquad y \geq 0$$

   is empty.

5. In a certain factory, say there are three machines M_1, M_2, M_3 used in making two products P_1, P_2. One unit of P_1 occupies M_1 for 5 minutes, M_2 for 3 minutes, and M_3 for 4 minutes. The corresponding data for one unit of P_2 are: M_1 for 1 minute, M_2 for 4 minutes, and M_3 for 3 minutes.

   The net profit per unit of P_1 produced is 30 dollars, and for P_2 20 dollars. The question is: what production plan gives the most profit?

   Suppose that x_1 units of P_1 and x_2 units of P_2 are produced per hour.

   a) If you want to maximize the profits, what function $F(x_1, x_2)$ are you maximizing?
   b) Since an hour has 60 minutes, the capacities of the three machines each impose a constraint on x_1 and x_2. What are these three constraints?
   c) Draw a graph of the x_1-x_2 plane showing the constraints and the feasible region (a pentagon).
   d) Compute the maximum profit per hour in this example.

6. [*Transportation Problem*] Suppose that Texas, California, and Alaska each produce a million barrels of oil:

   - 800,000 barrels are needed in Chicago at a distance of 1000, 2000, and 3000 miles from the three producers, respectively.
   - 2,200,000 barrels are needed in New England 1500, 3000, and 3700 miles away.

   If shipments cost one unit for each barrel-mile, what linear programming problem with five equality constraints (*if needed, introduce slack variables*) must be used to minimize the shipping costs?

7. Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be linear map (so it is represented by an $n \times k$ matrix).

   a) Show that the matrices A^*A and AA^* both have the same *non-zero* eigenvalues.

   b) If $\vec{v}_1$ and $\vec{v}_2$ are orthogonal eigenvectors of A^*A corresponding to non-zero eigenvalues, and if $\vec{u}_j = A\vec{v}_j$, show that the $\vec{u}_j$ are also orthogonal. [Noted later: Alas, this i1s a repeat of Homework Set 9, #4].

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 Let P_1, P_2,..., P_k be k points (think of them as *data*) in $\mathbb{R}^2$ and let $\mathcal{S}$ be the line

$$\mathcal{S} := \left\{ X \in \mathbb{R}^2 \,:\, \langle X,\, N\rangle = c \right\},$$

where $N \neq 0$ is a unit vector normal to the line and c is a real constant.

This problem outlines how to find the line that *best approximates the data points* in the sense that it minimizes the function

$$Q(N, c) := \sum_{j=1}^{k} \text{distance}\,(P_j, \mathcal{S})^2.$$

Determining this line means finding N and c.

a) Show that for a given point P, then

$$\text{distance}\,(P, \mathcal{S}) = |\langle P - X,\, N\rangle| = |\langle P,\, N\rangle - c\,|,$$

where X is any point in $\mathcal{S}$

b) First do the special case where the center of mass $\overline{P} := \frac{1}{k}\sum_{j=1}^{k} P_j$ is at the origin, so $\overline{P} = 0$. Show that for any P, then $\langle P,\, N\rangle^2 = \langle N,\, PP^*N\rangle$. Here view P as a column vector so PP^* is a 3×3 matrix.

Use this to observe that the desired line $\mathcal{S}$ is determined by letting N be an eigenvector of the matrix

$$A := \sum_{j=1}^{k} P_j P_j^T$$

corresponding to it's lowest eigenvalue. What is c in this case?

c) Reduce the general case to the previous case by letting $V_j = P_j - \overline{P}$.

d) Find the equation of the line $ax + by = c$ that, in the above sense, best fits the data points $(-1, 3)$, $(0, 1)$, $(1, -1)$, $(2, -3)$.

[Last revised: April 23, 2014]