ODE: Complex Eigenvalues

Say you want to solve the vector differential equation

$$X'(t) = AX, \qquad \text{where} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1}$$

If the eigenvalues of A (and hence the eigenvectors) are real, one has an idea how to proceed. However if the eigenvalues are complex, it is less obvious how to find the real solutions. Because we are interested in a real solution, we need a strategy to untangle this. We examine the case where A has complex eigenvalues $\lambda_1 = \lambda$ and $\lambda_2 = \bar{\lambda}$ with corresponding complex eigenvectors $W_1 = W$ and $W_2 = \overline{W}$.

The KEY OBSERVATION is that if $X(t)$ is a complex solution, split X in its real and imaginary parts, say $X(t) = U(t) + iV(t)$, where $U(t)$ and $V(t)$ are both real vectors. Then

$$\frac{dX}{dt} = \frac{dU}{dt} + i\frac{dV}{dt} \qquad \text{and} \qquad AX = AU + iAV.$$

Thus $U(t)$ and $V(t)$ are both *real* solutions of equation (1).

We apply this observation to our example. For the moment we ignore that in our special case λ_2 and W_2 are conjugates of λ_1 and W_1.

Since these W_1 and W_2 are linearly independent, we seek any solution as a linear combination,

$$X(t) = z_1(t)W_1 + z_2(t)W_2. \tag{2}$$

Clearly

$$X'(t) = z_1'(t)W_1 + z_2'(t)W_2$$

and

$$AX = z_1(t)AW_1 + z_2(t)AW_2 = z_1(t)\lambda_1 W_1 + z_2(t)\lambda_2 W_2.$$

Since we want $X' = Ax$, comparing the right sides of the equations above we find

$$z_1' = \lambda_1 z_1 \qquad \text{and} \qquad z_2' = \lambda_2 z_2.$$

These equations are uncoupled and easy to solve. Their solutions are

$$z_1(t) = c_1 e^{\lambda_1 t} \qquad \text{and} \qquad z_2(2) == c_2 e^{\lambda_2 t}$$

for any constants c_1 and c_2. Using equation (2) we see that for any constants c_1 and c_2

$$X(t) = c_1 e^{\lambda_1 t} W_1 + c_2 e^{\lambda_2 t} W_2$$

is a solution of equation (1). One often calls the $e^{\lambda_j t} W_j$ *eigensolutions* of the differential equation

We apply this to an example of our complex case where λ_2 and W_2 are complex conjugates of $\lambda_1 = \lambda$ and $W_1 = W$ and conclude that

$$X(t) = c_1 e^{\lambda t} W + c_2 e^{\overline{\lambda} t} \overline{W}. \tag{3}$$

By the KEY OBSERVATION made near the top of this page, the real and imaginary parts of the complex eigensolution $e^{\lambda t} W$ will be the desired real solutions.

EXAMPLE Let $A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$. By a routine computation the eigenvalues are $\lambda_1 = \lambda = 3 + i$ and $\lambda_2 = \overline{\lambda} = 3 - i$ with corresponding eigenvectors $W = W_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $W_2 = \overline{W} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. From equation (3) we know that $e^{\lambda t} W$ is a solution. By Euler's formula $e^{x+iy} = e^x(\cos y + i \sin y)$ and then taking real and imaginary parts we find

$$e^{\lambda t} W = e^{(3+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} = (e^{3t} \cos t + i e^{3t} \sin t) \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} e^{3t} \cos t \\ e^{3t} \sin t \end{pmatrix} + i \begin{pmatrix} e^{3t} \sin t \\ -e^{3t} \cos t \end{pmatrix}.$$

The last two vectors on the right are the two real solutions we were seeking. Thus, with this A, the most general real solution of $X' = AX$ is

$$X(t) = C_1 \begin{pmatrix} e^{3t} \cos t \\ e^{3t} \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \sin t \\ -e^{3t} \cos t \end{pmatrix}$$

with any real constants c_1 and c_2.
Note that repeating this process using the second term in equation (3) does not give us anything new because, except for the constants c_1 and c_2, this term is the complex conjugate of the fist term.

The identical procedure works for the more general case of equation (3) and shows that the most general real solution there is by taking any linear combination of the real and imaginary parts of $e^{\lambda t}W$.

The next example (taken from *Applied Linear Algebra* by Olver and Shakiban) uses the identical ideas.

EXAMPLE Solve the initial value problem

$$\begin{aligned} x_1' &= x_1 + 2x_2 & \qquad x_1(0) &= 2 \\ x_2' &= x_2 - 2x_3 & x_2(0) &= -1. \\ x_3' &= 2x_1 + 2x_2 - x_3 & x_3(0) &= -2 \end{aligned}$$

The coefficient matrix is

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Its eigenvalues and corresponding eigenvectors are

$$\lambda_1 = -1, \quad \lambda_2 = 1+2i, \quad \lambda_3 = 1-2i, \qquad W_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}, \quad W_3 = \begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}.$$

The corresponding eigensolutions $Y_j(t) = e^{\lambda_j t}W_j$ are

$$Y_1(t) = e^{-t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad Y_2(t) = e^{(1+2i)t}\begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}, \quad Y_3(t) = e^{(1-2i)t}\begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}.$$

As before, notice that $Y_3 = \overline{Y_2}$. One real solution is evidently Y_1. To get the other two, we take the real and imaginary parts of Y_2 (or equivalently, of Y_3). The process is routine:

$$Y_2(t) = e^{(1+2i)t} \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix} = (e^t \cos 2t + ie^t \sin 2t) \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} e^t \cos 2t \\ -e^t \sin 2t \\ e^t \cos 2t \end{pmatrix} + i \begin{pmatrix} e^t \sin 2t \\ e^t \cos 2t \\ e^t sin 2t \end{pmatrix}.$$

The three real linearly independent solutions are

$$X_1(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} e^t \cos 2t \\ -e^t \sin 2t \\ e^t \cos 2t \end{pmatrix}, \qquad X_3(t) = \begin{pmatrix} e^t \sin 2t \\ e^t \cos 2t \\ e^t sin 2t \end{pmatrix}.$$

and the general real solution is

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + c_3 X_3(t)$$

for any real constants c_1, c_2, c_3. We pick these constants to match the initial conditions

$$c_1 X_1(0) + c_2 X_2(0) + c_3 X_3(0) = X(0),$$

that is,

$$\begin{pmatrix} -c_1 + c_2 \\ c_1 + c_3 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}.$$

One finds that $c_1 = -2$, $c_2 = -0$, $c_3 = 1$. Thus the final solution is

$$X(t) = \begin{pmatrix} 2e^{-t} + e^t \sin 2t \\ -2e^{-t} + e^t \cos 2t \\ -2e^{-t} + e^t \sin 2t \end{pmatrix}.$$

[Last revised: April 26, 2014]