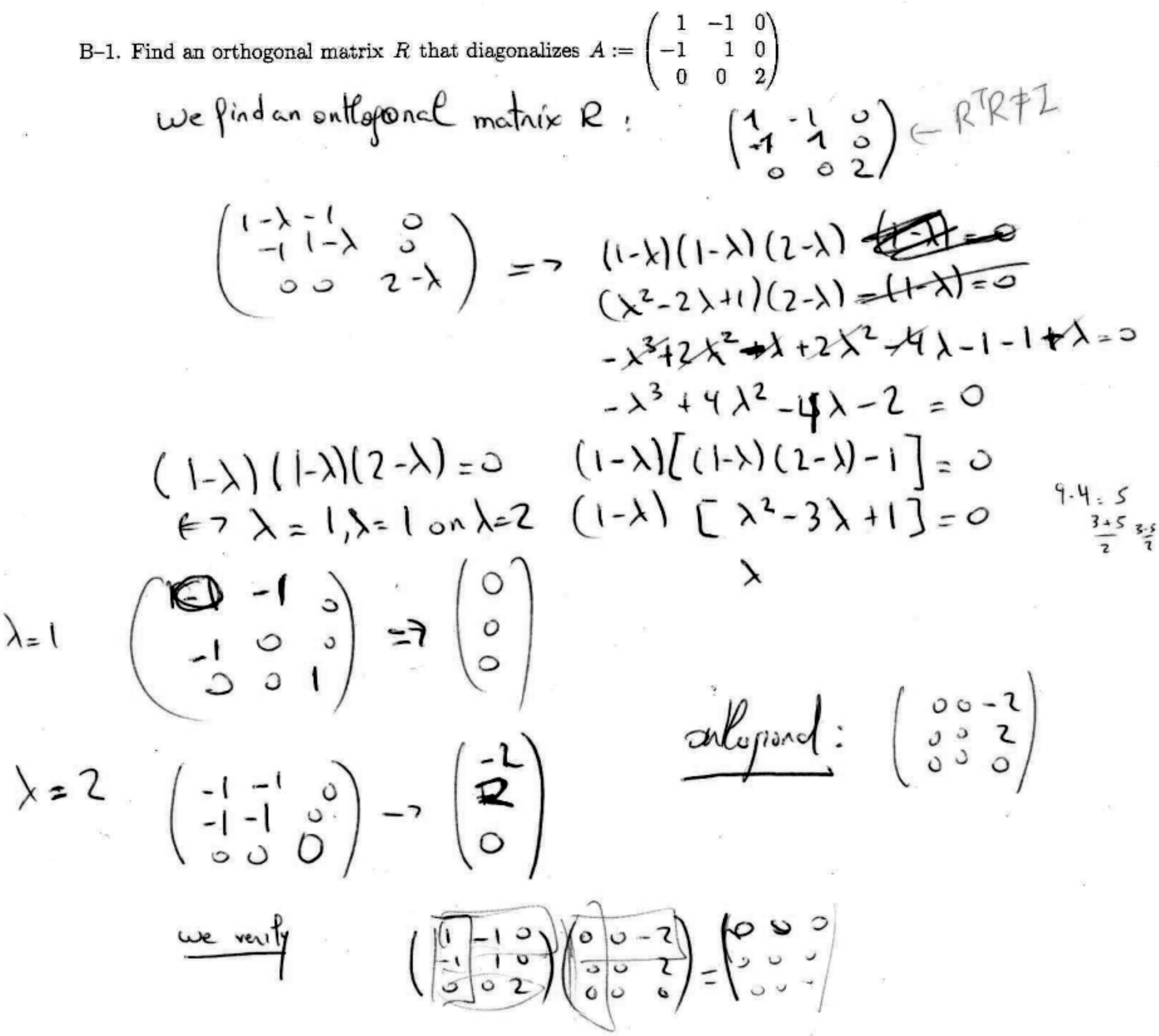
Math 312, Spring 2014 Jerry L. Kazdan

Example: Poor Exam Solution

This is an example of a very poor solution to an exam problem from some time in the past. The student showed no real understanding of the material.



B–1. Find an orthogonal matrix R that diagonalizes $A := \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

we find an orthogonal matrix R : $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \leftarrow R^T R \neq I$

$$\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} \Rightarrow (1-\lambda)(1-\lambda)(2-\lambda)\cancel{-(1-\lambda)} = 0$$

$$(\lambda^2-2\lambda+1)(2-\lambda)\cancel{-(1-\lambda)} = 0$$

$$-\lambda^3+2\lambda^2-\lambda+2\lambda^2-4\lambda-1-1+\lambda = 0$$

$$-\lambda^3+4\lambda^2-4\lambda-2 = 0$$

$(1-\lambda)(1-\lambda)(2-\lambda) = 0$ $\quad (1-\lambda)[(1-\lambda)(2-\lambda)-1] = 0$

$\Leftrightarrow \lambda = 1, \lambda = 1$ or $\lambda = 2$ $\quad (1-\lambda)[\lambda^2-3\lambda+1] = 0$

λ

$9-4=5$ $\quad \frac{3+\sqrt5}{2}$ $\frac{3-\sqrt5}{2}$

$\lambda = 1$ $\quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

orthogonal: $\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$\lambda = 2$ $\quad \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$

we verify $\quad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Given the matrix A, the first step, writing $A - \lambda I$ and then computing its determinant, was correct – but the computation was incorrect. That, in itself, was not terrible. It led him to the eigenvalues $\lambda = 1$, 1 and 2. That $\lambda = 1$ is incorrect while $\lambda = 2$ is actually a double root. He had several opportunities to catch this error, but never noticed them. Here are some details.

He should have noticed the error immediately when he found that corresponding to his $\lambda = 1$ his eigenvector was $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The zero vector is never an eigenvector [the definition of an eigenvalue and eigenvector is that there is a *nonzero* vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$]. Also, since his $\lambda = 1$ was a double root and the matrix is symmetric (so it is diagonalizable) there should have been two linearly independent eigenvectors for this eigenvalue.

His computation of the eigenvector corresponding to $\lambda = 2$ was also incorrect. He should have found that any vector of the form $\begin{pmatrix} a \\ -a \\ c \end{pmatrix}$ is an eigenvector, so the eigenspace has dimension 2. In turn, this should have made him realize that the eigenvalue $\lambda = 2$ is a *double* root of the characteristic polynomial, contrary to his earlier (incorrect) computation. – and gave him another opportunity to spot and correct his error.

He also could have noticed that the first two columns of A are linearly dependent, giving that $\ker A \neq 0$ so $\lambda = 0$ is an eigenvalue.

Finally, the matrix $R = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ is obviously not an orthogonal matrix. For instance, it is clearly not even invertible. His bottom line, "We verify" is simply absurd. What is he verifying? Moreover, the matrix multiplication to get the last column is not correct.

Bottom line: Computational errors are not big sins, but major conceptual errors are. He did not understand enough to know whan being an eigenvalue means nor what an orthogonal matrix is. This problem was intended to be a gift, an exercise that one should have learned how to solve in Math 240.

This example is almost a parody of how to solve the problem.

[Last revised: April 3, 2014]