Math 312 **Final Exam** Jerry L. Kazdan
May 5, 2014 12:00 – 2:00

DIRECTIONS This exam has three parts. Part A has 5 shorter questions, (6 points each), Part B has 6 True/False questions (5 points each), and Part C has 5 standard problems (12 points each). Maximum score is thus 120 points.
Closed book, no calculators, cell phones, or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

PART A: Five short answer questions (6 points each, so 30 points).

A–1. Suppose $T : \mathbb{R}^6 \to \mathbb{R}^4$ is a linear map represented by a matrix, A.

a) What are the possible values for the rank of A? Why?

SOLUTION By the Rank-Nullity Theorem $0 \le \operatorname{rank}(A) \le \min\{6, 4\} = 4$.

b) What are the possible values for the dimension of the kernel of A? Why?

SOLUTION Since $\dim(\operatorname{image}(A)) \le 4$, by the Rank-Nullity Theorem $2 \le \dim \ker(A) \le 6$.

c) Suppose the rank of A is as large as possible. What is the dimension of $\ker(A)^\perp$? Explain.

SOLUTION Since then $\dim(\operatorname{image}(A)) = 4$, then $\dim(\ker(A)) = 2$ so $\dim(\ker(A))^\perp = 6 - 4 = 2$.

A–2. In the following equations

$$\begin{aligned} x_1 + x_2 + \ 2x_3 + x_4 &= \ 1 \\ x_1 - x_2 - \ 2x_3 + x_4 &= \ 0 \\ -x_1 + x_2 - \ 2x_3 + x_4 &= \ 3 \\ -x_1 - x_2 + \ 2x_3 + x_4 &= \ 2 \end{aligned}$$

solve for for x_2 (only!). [OBSERVE that if you write this as $x_1\vec{v}_1 + \cdots + x_4\vec{v}_4 = \vec{b}$, then the vectors $\vec{v}_j$ are orthogonal.]

SOLUTION Take the inner product of $x_1\vec{v}_1 + \cdots + x_4\vec{v}_4 = \vec{b}$ with $\vec{v}_2$ to find

$$x_2\langle \vec{v}_2, \vec{v}_2\rangle = \langle \vec{b}, \vec{v}_2\rangle.$$

That is, $4x_2 = 2$ so $x_2 = 1/2$.

A–3. Let $P_1 = (a_1, b_1)$, $P_2 = (a_2, b_2)$, $\dots P_5 = (a_5, b_5)$ be five points in the plane $\mathbb{R}^2$. Find the point $Q = (x, y)$ that minimizes

$$f(x, y) = \|P_1 - Q\|^2 + \|P_2 - Q\|^2 + \cdots + \|P_5 - Q\|^2.$$

SOLUTION METHOD 1. Expand $f(x, y)$ to find

$$f(x, y) = [(a_1 - x)^2 + (b_1 - y)^2] + [(a_2 - x)^2 + (b_2 - y)^2] + \cdots + [(a_5 - x)^2 + (b_5 - y)^2].$$

At a minimum, the first partial derivatives are zero:

$$0 = f_x(x, y) = -2[(a_1 - x) + (a_2 - x) + \ldots + (a_5 - x)]$$

and

$$0 = f_y(x, y) = -2[(b_1 - y) + (b_2 - y) + \ldots + (b_5 - y)]$$

so

$$x = \frac{a_1 + a_2 + \cdots + a_5}{5} \qquad \text{and} \qquad y = \frac{b_1 + b_2 + \cdots + b_5}{5}$$

METHOD 1′. Same, but not using coordinates. Say f is minimized at Q. Then for any vector V, the function $\varphi(t) := f(Q + tV) = \sum_{j=1}^{5} \|P_j - (Q + tV)\|^2$ has a min at $t = 0$. Therefore $\varphi'(0) = 0$. Since

$$\frac{d}{dt}\|P_j - (Q + tV)\|^2\Big|_{t=0} = -2\langle P_j - Q,\, V\rangle$$

then

$$0 = -2\sum_{j=1}^{5}\langle P_j - Q,\, V\rangle = -2\langle P_1 + P_2 + \cdots + P_5 - 5Q,\, V\rangle.$$

Since this must hold for all V, then $P_1 + P_2 + \cdots + P_5 - 5Q = 0$, that is

$$Q = \frac{P_1 + P_2 + \cdots + P_5}{5}.$$

METHOD 2 This approach is clearer with n points $P_1, P_2, \ldots, P_n$. Since

$$\|P_j - Q\|^2 = \|P_j\|^2 - 2\langle P_j,\, Q\rangle + \|Q\|^2,$$

then, letting $\overline{P} = \frac{1}{n}(P_1 + P_2 + \cdots + P_n)$, we have

$$\begin{aligned} f(Q) = \sum_{j=1}^{n}\|P_j - Q\|^2 &= \left[\sum_{j=1}^{n}\|P_j\|^2\right] - 2n\langle \overline{P},\, Q\rangle + n\|Q\|^2 \\ &= \left[\sum_{j=1}^{n}\|P_j\|^2\right] + n\left[\|\overline{P} - Q\|^2 - \|\overline{P}\|^2\right] \end{aligned},$$

which is clearly minimized by letting $Q = \overline{P}$.

A–4. Let A be an $n \times k$ matrix.

a) If $\lambda_1 \neq 0$ is an eigenvalue of A^*A, show that it is also an eigenvalue of AA^*. [Note where you use $\lambda_1 \neq 0$].

SOLUTION Say $A^*A\vec{v}_1 = \lambda_1\vec{v}_1$ for some $\vec{v}_1 \neq 0$, Then

$$A(A^*A\vec{v}_1) = \lambda_1 A\vec{v}_1.$$

Let $\vec{w} = A\vec{v}_1$. Then $AA^*\vec{w} = \lambda_1\vec{w}$. Since $\lambda_1 \neq 0$, then $\vec{w} \neq 0$ so indeed $\vec{w}$ is an eigenvector of AA^* with eigenvalue λ_1.

b) If $\vec{v}_1$ and $\vec{v}_2$ are orthogonal eigenvectors of A^*A, let $\vec{u}_1 = A\vec{v}_1$, and $\vec{u}_2 = A\vec{v}_2$. Show that $\vec{u}_1$ and $\vec{u}_2$ are orthogonal.

SOLUTION $\langle\vec{u}_1,\, \vec{u}_2\rangle = \langle A\vec{v}_1,\, A\vec{v}_2\rangle = \langle\vec{v}_1,\, A^*A\vec{v}_2\rangle = \lambda_2\langle\vec{v}_1,\, \vec{v}_2\rangle = 0$.

A–5. Let A be a real matrix with the property that $\langle\vec{x},\, A\vec{x}\rangle = 0$ for all real vectors $\vec{x}$.

a) If A is a symmetric matrix, show this implies that $A = 0$.

SOLUTION Since A is a symmetric matrix, there is an orthonormal eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$ with $A\vec{v}_j = \lambda_j\vec{v}_j$. Writing $\vec{x} = y_1\vec{v}_1 + \cdots + y_n\vec{v}_n$ we have

$$0 = \langle\vec{x},\, A\vec{x}\rangle = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

for all $y_1,\, y_2, \ldots, y_n$. The only possibility is that all the $\lambda_j = 0$, that is, $A = 0$.

b) Give an example of a real matrix $A \neq 0$ that satisfies $\langle\vec{x},\, A\vec{x}\rangle = 0$ for all real vectors $\vec{x}$.

SOLUTION Let $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ be a rotation by 90 degrees.

PART B Six **True or False** questions (5 points each, so 30 points). Be sure to give a brief explanation.

B–1. If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a collection of vectors in $\mathbb{R}^5$, then the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ must be a three-dimensional subspace of $\mathbb{R}^5$.

SOLUTION False. The dimension is *at most* three.

B–2. The set of polynomials in $\mathcal{P}_4$ satisfying $p(0) = 2$ is a linear subspace of $\mathcal{P}_4$.

SOLUTION False. This set does not have the zero vector.

B–3. If $A : \mathbb{R}^k \to \mathbb{R}^n$ be a linear map and $\ker A^* = 0$, then for any $\vec{b} \in \mathbb{R}^n$ there is at least one solution of $A\vec{x} = \vec{b}$.

SOLUTION True since then $\text{image}\,(A) = (\ker A^*)^\perp$, which is everything.

B–4. If A is a 3×3 matrix with eigenvalues 1, 2, and 4, then $A - 4I$ is invertible.

SOLUTION False. The eigenvector corresponding to the eigenvalue $\lambda = 4$ is in the kernel of $A - 4I$.

B–5. If A is diagonalizable square matrix, then so is A^2.

SOLUTION True. Since A is diagonalizable, then for some invertible matrix S and a diagonal matrix D we have $A = SDS^{-1}$. But then $A^2 = SD^2S^{-1}$.

B–6. If a real matrix A can be orthogonally diagonalized, then it is self-adjoint (that is, symmetric).

SOLUTION True since $A = RDR^{-1}$ for some orthogonal matrix R. But $R^{-1} = R^*$ so

$$A^* = (RDR^*)^* = (R^*)^* DR^* = RDR^* = A.$$

PART C Five questions, 12 points each (so 60 points total).
[**Check** your computation of any eigenvalues by computing the trace and determinant of the matrix].

C–1. Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be a linear map.

a) If $k = n$, so A is represented by a *square* matrix, show that $\ker A = 0$ implies that A is also onto – and hence invertible.

SOLUTION Here $A : \mathbb{R}^n \to \mathbb{R}^n$. By the Rank-Nullity theorem, $\dim(\text{image}\,(A)) = n$ so the image of A is all of $\mathbb{R}^n$. Consequently A is onto and hence invertible.

b) If $k \neq n$, show that A *cannot* be invertible. NOTE there are two cases: $k < n$ and $k > n$.

SOLUTION If $k < n$ then by the Rank-Nullity theorem, the image of A is at most k so the map cannot be onto.

If $k > n$ then the image of A has dimension at most n so by the Nank-Nullity theorem $\dim(\ker(A)) \ge k - n > 0$.

C–2. a) Find an *orthogonal* matrix R that diagonalizes $A := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

SOLUTION We first find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) =& (3-\lambda)[(2-\lambda)^2 - 1] \\ =& (3-\lambda)(1-\lambda)(3-\lambda) \end{aligned}$$

so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 3$. By a routine computation $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ while $\vec{v}_2$ and $\vec{v}_3$ must both have the form $\begin{pmatrix} a \\ -a \\ c \end{pmatrix}$. One orthogonal set is $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. For the orthogonal matrix R we need *unit* orthogonal eigenvectors as columns, so

$$R = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{also} \qquad D = \begin{pmatrix} 1 & & \\ & 3 & \\ & & 3 \end{pmatrix}.$$

Then $A = RDR^*$. Note here, by chance, $R^* = R$.

b) Compute A^{50}.

SOLUTION

$$A^{50} = RD^{50}R^* = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 3^{50} & \\ & & 3^{50} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

C–3. Of the following four matrices, which can be orthogonally diagonalized; which can be diagonalized (but not orthogonally); and which cannot be diagonalized at all. Identify these – *fully explaining your reasoning.*

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

SOLUTION Because A and D are symmetric matrices, they can both be orthogonally diagonalized.

B is upper-triangular so its eigenvalues are on the diagonal. Since these three eigenvalues are *distinct*, it can be diagonalized. Since B is not symmetric, it cannot be orthogonally diagonalized

C is also upper-triangular so its eigenvalues are all 2. If C could be diagonalized, then it would be similar to $2I$, so $C = S(2I)S^{-1} = 2I$ for some invertible S. Since $C \neq 2I$, it cannot be diagonalized.

C–4. Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & -1 \end{pmatrix}$. Find a vector $\vec{v}$ that maximize $\|A\vec{x}\|$ on the unit disk $\|\vec{x}\| = 1$. What is this maximum value?

SOLUTION Note $\|A\vec{x}\|^2 = \langle A\vec{x},\, A\vec{x}\rangle = \langle \vec{x},\, A^*A\vec{x}\rangle$. Let $C := A^*A$. It is a symmetric positive semi-definite symmetric matrix (in fact, this C is positive definite). To maximize $\|A\vec{x}\|$ we pick $\vec{x}$ to be an eigenvector of C corresponding to its largest eigenvalue.

Now $C = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$. Its eigenvalues are $\lambda_1 = 9$ and $\lambda_2 = 1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Thus to maximize $\|A\vec{x}\|$ we let $\vec{x}$ be a *unit* vector in the direction of $\vec{v}_1$, so $\vec{x} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$. Then $\|A\vec{x}\| = \sigma_1 = \sqrt{\lambda_1} = 3$.

C–5. Let $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be a solution of the system of differential equations

$$\begin{aligned} x_1' &= \ \ cx_1 + x_2 \\ x_2' &= -x_1 + cx_2 \end{aligned}.$$

For which value(s) of the real constant c do *all* solutions $\vec{x}(t)$ converge to 0 as $t \to \infty$?

SOLUTION Rewrite this as $\vec{x}'(t) = A\vec{x}$, where $A = \begin{pmatrix} c & 1 \\ -1 & c \end{pmatrix}$. By a routine computation the eigenvalues are $\lambda_1 = c + i$ and $\lambda_2 = c - i$. Since these are distinct, we can diagonalize A. Say the corresponding eigenvectors are $\vec{v}_1$ and $\vec{v}_2 (= \overline{\vec{v}_1})$. We could compute them easily – but won't since we will not need them explicitly.

Since the $\vec{v}_j$ are a basis for $\mathbb{R}^2$ we can write

$$\vec{x}(t) = y_1(t)\vec{v}_1 + y_2(t)\vec{v}_2, \qquad (1)$$

where the coefficients $y_j(t)$ are to be found. Now

$$\vec{x}\,'(t) = y_1'(t)\vec{v}_1 + y_2'(t)\vec{v}_2 \qquad \text{and} \qquad A\vec{x}(t) = \lambda_1 y_1(t)\vec{v}_1 + \lambda_2 y_2(t)\vec{v}_2.$$

Because $\vec{x}\,' = A\vec{x}$, comparing these we see that

$$y_1' = \lambda_1 y_1 \qquad \text{and} \qquad y_2' = \lambda_2 y_2$$

whose solutions are

$$y_1(t) = ae^{\lambda_1 t} = ae^{(c+i)t} = ae^{ct}(\cos t + i \sin t)$$

and

$$y_2(t) = be^{\lambda_2 t} = be^{(c-i)t} = be^{ct}(\cos t - i \sin t),$$

where a and b can be any (complex) constants. In equation (1), because $\vec{v}_1$ and $\vec{v}_2$ are constant vectors, for all solutions $\vec{x}(t) \to 0$ as $t \to \infty$, we need that the $|y_j(t)| \to 0$. But since $|\cos t \pm i \sin t| = 1$, then

$$|y_j(t)| = e^{ct}|\cos t \pm i \sin t| = e^{ct}.$$

Because c is a real number, $e^{ct} \to 0$ as $t \to \infty$ if (and only if) $c < 0$.