Math 312 — **Final Exam** — Jerry L. Kazdan

May 5, 2014 — 12:00 – 2:00

DIRECTIONS This exam has three parts. Part A has 5 shorter questions, (6 points each), Part B has 6 True/False questions (5 points each), and Part C has 5 standard problems (12 points each). Maximum score is thus 120 points.
Closed book, no calculators, cell phones, or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

PART A: Five short answer questions (6 points each, so 30 points).

A–1. Suppose $T: \mathbb{R}^6 \to \mathbb{R}^4$ is a linear map represented by a matrix, A.

a) What are the possible values for the rank of A? Why?

b) What are the possible values for the dimension of the kernel of A? Why?

c) Suppose the rank of A is as large as possible. What is the dimension of $\ker(A)^{\perp}$? Explain.

A–2. In the following equations

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 1 \\ x_1 - x_2 - 2x_3 + x_4 &= 0 \\ -x_1 + x_2 - 2x_3 + x_4 &= 3 \\ -x_1 - x_2 + 2x_3 + x_4 &= 2 \end{aligned}$$

solve for for x_2 (only!). [OBSERVE that if you write this as $x_1\vec{v}_1 + \cdots + x_4\vec{v}_4 = \vec{b}$, then the vectors $\vec{v}_j$ are orthogonal.]

A–3. Let $P_1 = (a_1, b_1)$, $P_2 = (a_2, b_2)$, $\ldots$ $P_5 = (a_5, b_5)$ be five points in the plane $\mathbb{R}^2$. Find the point $Q = (x, y)$ that minimizes

$$f(x,y) = \|P_1 - Q\|^2 + \|P_2 - Q\|^2 + \cdots + \|P_5 - Q\|^2.$$

A–4. Let A be an $n \times k$ matrix.

a) If $\lambda_1 \neq 0$ is an eigenvalue of A^*A, show that it is also an eigenvalue of AA^*. [Note where you use $\lambda_1 \neq 0$].

b) If $\vec{v}_1$ and $\vec{v}_2$ are orthogonal eigenvectors of A^*A, let $\vec{u}_1 = A\vec{v}_1$, and $\vec{u}_2 = A\vec{v}_2$. Show that $\vec{u}_1$ and $\vec{u}_2$ are orthogonal.

A–5. Let A be a real matrix with the property that $\langle \vec{x}, A\vec{x} \rangle = 0$ for all real vectors $\vec{x}$.

a) If A is a symmetric matrix, show this implies that $A = 0$.

b) Give an example of a matrix $A \neq 0$ that satisfies $\langle \vec{x}, A\vec{x} \rangle = 0$ for all real vectors $\vec{x}$.

Part B Six **True or False** questions (5 points each, so 30 points). Be sure to give a brief explanation.

B–1. If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a collection of vectors in $\mathbb{R}^5$, then the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ must be a three-dimensional subspace of $\mathbb{R}^5$.

B–2. The set of polynomials in $\mathcal{P}_4$ satisfying $p(0) = 2$ is a linear subspace of $\mathcal{P}_4$.

B–3. If $A : \mathbb{R}^k \to \mathbb{R}^n$ be a linear map and $\ker A^* = 0$, then for any $\vec{b} \in \mathbb{R}^n$ there is at least one solution of $A\vec{x} = \vec{b}$.

B–4. If A is a 3×3 matrix with eigenvalues 1, 2, and 4, then $A - 4I$ is invertible.

B–5. If A is diagonalizable square matrix, then so is A^2.

B–6. If a real matrix A can be orthogonally diagonalized, then it is self-adjoint (that is, symmetric).

Part C Five questions, 12 points each (so 60 points total).
[**Check** your computation of any eigenvalues by computing the trace and determinant of the matrix].

C–1. Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be a linear map.

a) If $k = n$, so A is represented by a *square* matrix, show that $\ker A = 0$ implies that A is also onto – and hence invertible.

b) If $k \neq n$, show that A *cannot* be invertible. Note there are two cases: $k < n$ and $k > n$.

C–2. a) Find an *orthogonal* matrix R that diagonalizes $A := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

b) Compute A^{50}.

C–3. Of the following four matrices, which can be orthogonally diagonalized; which can be diagonalized (but not orthogonally); and which cannot be diagonalized at all. Identify these – *fully explaining your reasoning.*

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

C–4. Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & -1 \end{pmatrix}$. Find a vector $\vec{v}$ that maximize $\|A\vec{x}\|$ on the unit disk $\|\vec{x}\| = 1$. What is this maximum value?

C–5. Let $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be a solution of the system of differential equations

$$\begin{aligned} x_1' &= cx_1 + x_2 \\ x_2' &= -x_1 + cx_2 \end{aligned}.$$

For which value(s) of the real constant c do *all* solutions $\vec{x}(t)$ converge to 0 as $t \to \infty$?