Math 312 | **Exam 2** | Jerry L. Kazdan
April 1, 2014 | | 9:00 – 10:20

Directions This exam has two parts. Part A has 4 shorter questions, (5 points each so total 20 points) while Part B had 6 problems (12 points each, so total is 72 points). Maximum score is thus 92 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

Part A: Four short answer questions (5 points each, so 20 points).

A–1. Let A be a 3×3 real matrix two of whose eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 1 - 2i$, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, what are λ_3 and $\mathbf{v}_3$?

Solution We know that complex eigenvalues come in pairs i.e. $\lambda_3 = \overline{\lambda_2} = 1 + 2i$ and $A\overline{\mathbf{v}_2} = \overline{A\mathbf{v}_2} = \overline{\lambda_2}\overline{\mathbf{v}_2}$ hence $\mathbf{v}_3 = \overline{\mathbf{v}_2}$.

A–2. Given a *unit* vector $\mathbf{w} \in \mathbb{R}^n$, let $W = \operatorname{span}\{\mathbf{w}\}$ and consider the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T(\mathbf{x}) = 2\operatorname{Proj}_W(\mathbf{x}) - \mathbf{x},$$

where $\operatorname{Proj}_W(\mathbf{x})$ is the orthogonal projection onto W. Show that T is one-to-one.

Method 1 We need to show that the kernel of T is trivial, so we need to solve:

$$2\operatorname{Proj}_W(\mathbf{x}) - \mathbf{x} = 0 \tag{1}$$

To the above equation we apply T again and obtain:

$$0 = T(2\operatorname{Proj}_W(\mathbf{x}) - \mathbf{x}) = 2\operatorname{Proj}_W(2\operatorname{Proj}_W(\mathbf{x}) - \mathbf{x}) - 2\operatorname{Proj}_W(\mathbf{x}) + \mathbf{x}$$

so:

$$0 = 4\operatorname{Proj}_W(\mathbf{x}) - 2\operatorname{Proj}_W(\mathbf{x}) - 2\operatorname{Proj}_W(\mathbf{x}) + \mathbf{x} = \mathbf{x}$$

Hence, the kernel of T is trivial, namely T is one-to-one.

Method 2 Since $\mathbf{w}$ is a unit vector, $\operatorname{Proj}_W(\mathbf{x}) = \langle \mathbf{x}, \mathbf{w}\rangle\mathbf{w}$ so equation (1) is

$$2\langle \mathbf{x}, \mathbf{w}\rangle\mathbf{w} = \mathbf{x}.$$

Taking the inner product of this with $\mathbf{w}$ gives $2\langle \mathbf{x}, \mathbf{w}\rangle = \langle \mathbf{x}, \mathbf{w}\rangle$ so $\langle \mathbf{x}, \mathbf{w}\rangle = 0$. Equation (1) then gives $\mathbf{x} = 0$.

Method 3 Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be any projection, not necessarily orthogonal. It has the property $P^2 = P$. Define

$$T\mathbf{x} := cP\mathbf{x} - \mathbf{x}$$

for any constant c. Claim: if $c \neq 1$, then $\ker T = 0$ (so T is one-to-one). To see this, apply P to both sides of $cP\mathbf{x} = \mathbf{x}$ and use $P^2 = P$ to find $cP\mathbf{x} = P\mathbf{x}$. Because $c \neq 1$, then $P\mathbf{x} = 0$. Consequently $\mathbf{x} = 0$.

A–3. Let A be an invertible matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. What can you say about the eigenvalues and eigenvectors of A^{-1}? Justify your response.

SOLUTION Since A invertible we have that $A\vec{v}_i = \lambda_i \vec{v}_i$ and $\lambda_i \neq 0$ for all i. Hence by multiplying $\frac{1}{\lambda_i}A^{-1}$ on both sides of $A\vec{v}_i = \lambda_i \vec{v}_i$ we obtain that $A^{-1}\vec{v}_i = \frac{1}{\lambda_i}\vec{v}_i$. So $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}$ are the eigenvalues of A^{-1} with the same corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.

A–4. Let A be an $n \times n$ real self-adjoint matrix and $\mathbf{v}$ an eigenvector with eigenvalue λ. Let $W = \text{span}\{\mathbf{v}\}$.

a) If $\mathbf{w} \in W$, show that $A\mathbf{w} \in W$

SOLUTION If $\mathbf{w} \in W$ then $\mathbf{w} = k\mathbf{v}$. Hence $A\mathbf{w} = Ak\mathbf{v} = k\lambda\mathbf{v} \in W$.

b) If $\mathbf{z} \in W^{\perp}$, show that $A\mathbf{z} \in W^{\perp}$.

SOLUTION If $\mathbf{z} \in W^{\perp}$ then $\langle \mathbf{z}, \mathbf{v} \rangle = 0$. Hence $\langle A\mathbf{z}, \mathbf{v} \rangle = \langle \mathbf{z}, A^*\mathbf{v} \rangle = \langle \mathbf{z}, A\mathbf{v} \rangle = \langle \mathbf{z}, \lambda\mathbf{v} \rangle = \lambda\langle \mathbf{z}, \mathbf{v} \rangle = 0$ so $A\mathbf{z} \in W^{\perp}$.

PART B Six questions, 12 points each (so 72 points total).

B–1. Let A be a real symmetric matrix. Say that $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to *distinct* eigenvalues $\lambda_1 \neq \lambda_2$. Show that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

SOLUTION We have that:

$$\lambda_1\langle \vec{v}_1, \vec{v}_2 \rangle = \langle A\vec{v}_1, \vec{v}_2 \rangle = \langle \vec{v}_1, A^*\vec{v}_2 \rangle = \langle \vec{v}_1, A\vec{v}_2 \rangle = \lambda_2\langle \vec{v}_1, \vec{v}_2 \rangle$$

$$(\lambda_1 - \lambda_2)\langle \vec{v}_1, \vec{v}_2 \rangle = 0$$

so $\langle \vec{v}_1, \vec{v}_2 \rangle = 0$, namely $\vec{v}_1, \vec{v}_2$ are orthogonal.

METHOD 2 Since $\lambda_1 \neq \lambda_2$, at least one of them is not zero. Say $\lambda_2 \neq 0$. Now use

$$\langle A\vec{v}_1, A\vec{v}_2 \rangle = \lambda_1\lambda_2\langle \vec{v}_1, \vec{v}_2 \rangle$$

and

$$\langle A\vec{v}_1, A\vec{v}_2 \rangle = \langle \vec{v}_1, A^2\vec{v}_2 \rangle = \lambda_2^2\langle \vec{v}_1, \vec{v}_2 \rangle.$$

Now use $\lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$ to conclude $\langle \vec{v}_1, \vec{v}_2 \rangle = 0$.

B–2. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs. One has data on which location the cars are returned daily:

- RENTED AT AIRPORT: 5% are returned to the City and 20% to the Suburbs. The rest are returned to the Airport.
- RENTED IN CITY : 10% are returned to Airport, 10% returned to Suburbs.
- RENTED IN SUBURBS: 20% are returned to the Airport and 5% to the City.

If initially there are 20 cars at the Airport, 65 in the city, and 15 in the suburbs, what is the long-term distribution of the cars?

Solution The equations we obtain from the information given is:

$$x_{k+1} = 0.75x_k + 0.1y_k + 0.2z_k$$

$$y_{k+1} = 0.05x_k + 0.8y_k + 0.05z_k$$

$$z_{k+1} = 0.2x_k + 0.1y_k + 0.75z_k$$

where x's, y's, z's correspond to information about cars rented at airport, city, suburbs respectively. Hence the transition matrix is:

$$T = \begin{pmatrix} 0.75 & 0.1 & 0.2 \\ 0.05 & 0.8 & 0.05 \\ 0.2 & 0.1 & 0.75 \end{pmatrix}$$

which is regular, so we need to find the probability eigenvector corresponding to the eigenvalue $\lambda = 1$. Solving $T\vec{v} = \vec{v}$ we obtain $v_1 = v_3$ and $v_2 = 0.5v_3$ where $\vec{v} = (v_1, v_2, v_3)$. Hence a eigenvector corresponding to $\lambda = 1$ is:

$$\vec{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

so the unique probability eigenvector corresponding to $\lambda = 1$ is:

$$1/5\vec{v} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix}.$$

Now, initially there were 100 cars so the long term distribution is: 40 cars at the Airport, 20 at the City and 40 at the Suburbs.

B–3. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$.

a) What is the dimension of the image of A? Why?

Solution Since $\operatorname{im} A$ is the column-space of A we have that $\operatorname{im} A = \operatorname{span}\{(1,1,1)\}$, so $\dim(\operatorname{im} A) = 1$.

b) What is the dimension of the kernel of A? Why?

Solution From rank-nullity theorem and part (a) we have that $\dim(\ker A) = 2$.

c) What are the eigenvalues of A? Why?

Solution 1: Since $\ker A$ is 2-dimensional it implies that two of the eigenvalues of A are 0. Also since the trace of A (which is equal to 4) is equal to the sum of its eigenvalues we have that the third eigenvalue is equal to 4.

Solution 2: Using the characteristic polynomial of A which is: $p_A(\lambda) = \lambda^2(4 - \lambda)$.

d) What are the eigenvalues of $B := \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 1 & 1 & 5 \end{pmatrix}$? Why? [HINT: $B = A + 3I$].

SOLUTION If λ is an eigenvalue of A and $\mathbf{v}$ the corresponding eigenvector then:

$$B\mathbf{v} = (A + 3I)\mathbf{v} = (\lambda + 3)\mathbf{v}$$

hence using part (c) we obtain that the eigenvalues of B are $3, 3, 7$.

B–4. For certain polynomials $\mathbf{p}(t)$, $\mathbf{q}(t)$, and $\mathbf{r}(t)$, say we are given the following table of inner products:

$\langle\, ,\, \rangle$	$\mathbf{p}$	$\mathbf{q}$	$\mathbf{r}$
$\mathbf{p}$	4	0	8
$\mathbf{q}$	0	1	0
$\mathbf{r}$	8	0	50

For example, $\langle \mathbf{q}, \mathbf{r}\rangle = \langle \mathbf{r}, \mathbf{q}\rangle = 0$. Let E be the span of $\mathbf{p}$ and $\mathbf{q}$.

a) Compute $\langle \mathbf{p}, \mathbf{q} + \mathbf{r}\rangle$.

SOLUTION $\langle \mathbf{p}, \mathbf{q} + \mathbf{r}\rangle = \langle \mathbf{p}, \mathbf{q}\rangle + \langle \mathbf{p}, \mathbf{r}\rangle = 0 + 8 = 8$

b) Compute $\|\mathbf{q} + \mathbf{r}\|$.

SOLUTION $\|\mathbf{q} + \mathbf{r}\| = \sqrt{\langle \mathbf{q}, \mathbf{q}\rangle + \langle \mathbf{r}, \mathbf{r}\rangle + 2\langle \mathbf{q}, \mathbf{r}\rangle} = \sqrt{1 + 50 + 0} = \sqrt{51}$

c) Find the orthogonal projection $\text{Proj}_E \mathbf{r}$. [Express your solution as linear combinations of $\mathbf{p}$ and $\mathbf{q}$.]

SOLUTION $\text{Proj}_E \mathbf{r} = \frac{\langle \mathbf{r}, \mathbf{p}\rangle}{\langle \mathbf{p}, \mathbf{p}\rangle}\mathbf{p} + \frac{\langle \mathbf{r}, \mathbf{q}\rangle}{\langle \mathbf{q}, \mathbf{q}\rangle}\mathbf{q} = 2\mathbf{p}$.

d) Find an orthonormal basis of the span of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$. [Express your results as linear combinations of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$.]

SOLUTION We apply the Gram-Schmidt process to first get an orthogonal basis $\{\mathbf{u}_1, Bu_2, Bu_3\}$ and then the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

$$\mathbf{u}_1 = \mathbf{q} \quad \text{and} \quad \mathbf{e}_1 = q$$

$$\mathbf{u}_2 = \mathbf{p} - \frac{\langle \mathbf{p}, \mathbf{q}\rangle}{\langle \mathbf{q}, \mathbf{q}\rangle}\mathbf{q} = \mathbf{p} \quad \text{and} \quad \mathbf{e}_2 = 1/2\mathbf{p}$$

$$\mathbf{u}_3 = \mathbf{r} - \frac{\langle \mathbf{r}, \mathbf{q}\rangle}{\langle \mathbf{q}, \mathbf{q}\rangle}\mathbf{q} - \frac{\langle \mathbf{r}, \mathbf{p}\rangle}{\langle \mathbf{p}, \mathbf{p}\rangle}\mathbf{p} = \mathbf{r} - 2\mathbf{p} \quad \text{and}$$

$$\mathbf{e}_3 = \frac{\mathbf{r} - 2\mathbf{p}}{\sqrt{34}} \quad \text{since} \quad \|\mathbf{r} - 2\mathbf{p}\|^2 = \langle \mathbf{r}, \mathbf{r}\rangle + 4\langle \mathbf{p}, \mathbf{p}\rangle - 4\langle \mathbf{r}, \mathbf{p}\rangle = 50 + 16 - 32 = 34.$$

B–5. An $n \times n$ matrix is called *nilpotent* if A^k equals the zero matrix for some positive integer k. (For instance, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is nilpotent.)

a) If λ is an eigenvalue of a nilpotent matrix A, show that $\lambda = 0$. (Hint: start with the equation $A\vec{x} = \lambda\vec{x}$.)

SOLUTION We have $A\vec{x} = \lambda\vec{x}$ so $A^k\vec{x} = \lambda^k\vec{x}$. Hence $\lambda^k\vec{x} = 0$ so $\lambda = 0$ since $\vec{x} \neq 0$ (because it is an eigenvector).

b) Show that if A is both nilpotent and diagonalizable, then A is the zero matrix. [Hint: use Part a).]

SOLUTION From part (a) we deduce that all eigenvalues of A are zero, Hence A is similar to the zero matrix hence $A = S(\mathbf{0})S^{-1} = \mathbf{0}$ where $\mathbf{0}$ the zero matrix and S some matrix.

c) Let A be the matrix that represents $T : \mathcal{P}_5 \to \mathcal{P}_5$ (polynomials of degree at most 5) given by differentiation: $Tp = dp/dx$. Without doing any computations, explain why A must be nilpotent.

SOLUTION Since p polynomial of degree at most 5 we have that T^6 is the zero map ($T^6 = T \circ T \circ T \circ T \circ T \circ T$ composition of T with itself) hence $A^6 = \mathbf{0}$ namely A nilpotent.

B–6. Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be a linear map. Show that

$$\dim(\ker A) - \dim(\ker A^*) = k - n.$$

In particular, for a square matrix, $\dim(\ker A) = \dim(\ker A^*)$.

SOLUTION 1: Since in $\mathbb{R}^k$, $(\operatorname{im} A^*)^\perp = \ker A$, we have that

$$\dim(\ker A) + \dim(\operatorname{im} A^*) = k$$

Also, since $A^* : \mathbb{R}^n \to \mathbb{R}^k$, by the rank-nullity theorem

$$\dim(\ker A^*) + \dim(\operatorname{im} A^*) = n$$

Then we subtract to obtain:

$$\dim(\ker A) - \dim(A^*) = k - n.$$

SOLUTION 2: Since $A^* : \mathbb{R}^n \to \mathbb{R}^k$, by a homework problem $\dim \operatorname{im} A = \dim \operatorname{im} A^*$. Using rank-nullity theorem we have:

$$\dim(\ker A) - \dim(\ker A^*) = (\dim \mathbb{R}^k - \dim \operatorname{im} A) - (\dim \mathbb{R}^n - \dim \operatorname{im} A^*) = k - n$$