Math 312 | Exam 2 | Jerry L. Kazdan
April 1, 2014 | | 9:00 – 10:20

DIRECTIONS This exam has two parts. Part A has 4 shorter questions, (5 points each so total 20 points) while Part B had 6 problems (12 points each, so total is 72 points). Maximum score is thus 92 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

PART A: Four short answer questions (5 points each, so 20 points).

A–1. Let A be a 3×3 real matrix two of whose eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 1 - 2i$, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, what are λ_3 and $\mathbf{v}_3$?

A–2. Given a *unit* vector $\mathbf{w} \in \mathbb{R}^n$, let $W = \operatorname{span}\{\mathbf{w}\}$ and consider the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T(\mathbf{x}) = 2\operatorname{Proj}_W(\mathbf{x}) - \mathbf{x},$$

where $\operatorname{Proj}_W(\mathbf{x})$ is the orthogonal projection onto W. Show that T is one-to-one.

A–3. Let A be an invertible matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. What can you say about the eigenvalues and eigenvectors of A^{-1}? Justify your response.

A–4. Let A be an $n \times n$ real self-adjoint matrix and $\mathbf{v}$ an eigenvector with eigenvalue λ. Let $W = \operatorname{span}\{\mathbf{v}\}$.

a) If $\mathbf{w} \in W$, show that $A\mathbf{w} \in W$

b) If $\mathbf{z} \in W^\perp$, show that $A\mathbf{z} \in W^\perp$.

PART B Six questions, 12 points each (so 72 points total).

B–1. Let A be a real symmetric matrix. Say that $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to *distinct* eigenvalues $\lambda_1 \neq \lambda_2$. Show that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

B–2. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs. One has data on which location the cars are returned daily:

- RENTED AT AIRPORT: 5% are returned to the City and 20% to the Suburbs. The rest are returned to the Airport.
- RENTED IN CITY : 10% are returned to Airport, 10% returned to Suburbs.
- RENTED IN SUBURBS: 20% are returned to the Airport and 5% to the City.

If initially there are 20 cars at the Airport, 65 in the city, and 15 in the suburbs, what is the long-term distribution of the cars?

B–3. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$.

a) What is the dimension of the image of A? Why?

b) What is the dimension of the kernel of A? Why?

c) What are the eigenvalues of A? Why?

d) What are the eigenvalues of $B := \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 1 & 1 & 5 \end{pmatrix}$? Why? [HINT: $B = A + 3I$].

B–4. For certain polynomials $\mathbf{p}(t)$, $\mathbf{q}(t)$, and $\mathbf{r}(t)$, say we are given the following table of inner products:

$\langle\, ,\, \rangle$	$\mathbf{p}$	$\mathbf{q}$	$\mathbf{r}$
$\mathbf{p}$	4	0	8
$\mathbf{q}$	0	1	0
$\mathbf{r}$	8	0	50

For example, $\langle \mathbf{q}, \mathbf{r} \rangle = \langle \mathbf{r}, \mathbf{q} \rangle = 0$. Let E be the span of $\mathbf{p}$ and $\mathbf{q}$.

a) Compute $\langle \mathbf{p}, \mathbf{q} + \mathbf{r} \rangle$.

b) Compute $\|\mathbf{q} + \mathbf{r}\|$.

c) Find the orthogonal projection $\mathrm{Proj}_E \mathbf{r}$. [Express your solution as linear combinations of $\mathbf{p}$ and $\mathbf{q}$.]

d) Find an orthonormal basis of the span of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$. [Express your results as linear combinations of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$.]

B–5. An $n \times n$ matrix is called *nilpotent* if A^k equals the zero matrix for some positive integer k. (For instance, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is nilpotent.)

a) If λ is an eigenvalue of a nilpotent matrix A, show that $\lambda = 0$. (Hint: start with the equation $A\vec{x} = \lambda\vec{x}$.)

b) Show that if A is both nilpotent and diagonalizable, then A is the zero matrix. [Hint: use Part a).]

c) Let A be the matrix that represents $T : \mathcal{P}_5 \to \mathcal{P}_5$ (polynomials of degree at most 5) given by differentiation: $Tp = dp/dx$. Without doing any computations, explain why A must be nilpotent.

B–6. Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be a linear map. Show that

$$\dim(\ker A) - \dim(\ker A^*) = k - n.$$

In particular, for a square matrix, $\dim(\ker A) = \dim(\ker A^*)$.