Math 312, Spring 2013 Jerry L. Kazdan

Problem Set 3

DUE: In class Thurs. Jan 31 [*Late papers will be accepted until 1:00 on Friday*].

1. Let A and B both be $n \times n$ matrices. What's wrong with the formula $(A+B)^2 = A^2 + 2AB + B^2$? Prove that if this formula is true for A and B, then A and B commute.

2. Which of the following subsets of $\mathbb{R}^2$ are actually subspaces? Explain.
   a) $\{(x,y) \mid xy = 0\}$
   b) $\{(x,y) \mid x$ and $y$are both integers$\}$
   c) $\{(x,y) \mid x+y = 0\}$
   d) $\{(x,y) \mid x+y = 2\}$
   e) $\{(x,y) \mid x+y \ge 0\}$

3. Let V and W be linear spaces and $T : V \to W$ a linear map.
   a) Assume the kernel of T is trivial, that is, the only solution of the homogeneous equation $T\vec{x} = 0$ is $\vec{x} = 0$. Prove that if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$.
   b) Conversely, if T has the property that "if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$," show that the kernel of T is trivial.

4. Say $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent vectors in $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map.
   a) Show by an example, say for $n = 2$, that $T\vec{v}_1, \ldots, T\vec{v}_n$ need not be linearly independent.
   b) However, show that if the kernel of T is trivial, then these vectors $T\vec{v}_1, \ldots, T\vec{v}_n$ are linearly independent.

5. Let $A : \mathbb{R}^3 \to \mathbb{R}^5$ and $B : \mathbb{R}^5 \to \mathbb{R}^2$.
   a) What are the maximum and minimum values for the dimension of the kernels of A, B, and BA?
   b) What are the maximum and minimum values for the dimension of the images of A, B, and BA?

6. [BRETSCHER, SEC. 2.4 #52]. Let $A := \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{pmatrix}$. Find a vector $\vec{b}$ in $\mathbb{R}^4$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, that is, it has no solution.

7. Find a real 2×2 matrix A (with $A^2 \neq I$ and $A^3 \neq I$) so that $A^6 = I$. For your example, is A^4 invertible?

8. Let A, B, and C be $n \times n$ matrices with A and C invertible. Solve the equation $ABC = I - A$ for B.

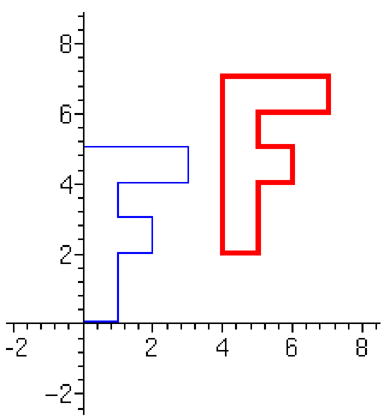
9. If a square matrix M has the property that $M^4 - M^2 + 2M - I = 0$, show that M is invertible. [SUGGESTION: . Find a matrix N so that $MN = I$. This is very short.]

10. Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

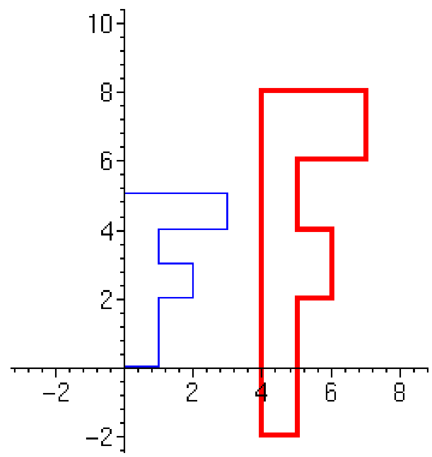
$$F(X) = V + AX,$$

where V is a vector. Note that $F(0) = V$.

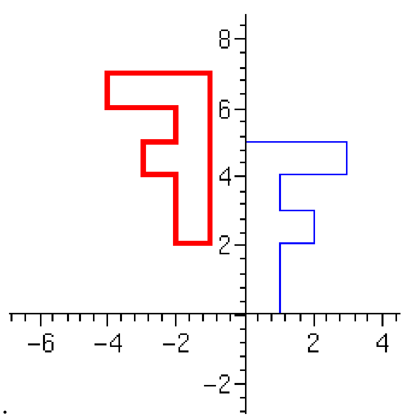
Find the vector V and the matrix A that describe each of the following mappings [here the light blue F is mapped to the dark red F].



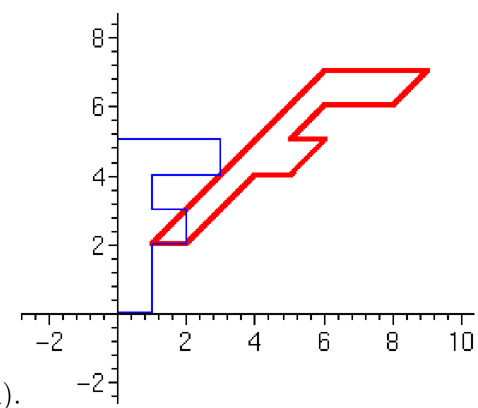
a).



b).



c).



d).

11. Let $\vec{e}_1 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^n$ and let $\vec{v}$ and $\vec{w}$ be any non-zero vectors in $\mathbb{R}^n$.
   a) Show there is an invertible matrix B with $B\vec{e}_1 = \vec{v}$.
   b) Show there is an invertible matrix M with $M\vec{w} = \vec{v}$.

12. [Like Bretscher, Sec. 2.4 #40]. Let A be a matrix, not necessarily square.
   a) If A has two equal rows, show that it is not onto (and hence not invertible).
   b) If A has two equal columns, show that it is not one-to-one (and hence not invertible).

[Last revised: January 30, 2014]