Rust Remover [Due: Never]

1. Solve the following system – or show that no solution exists:

$$\begin{aligned} x + 2y \qquad &= \ \ 1 \\ 3x + 2y + 4z &= \ \ 7 \\ -2x + y - 2z &= -1 \end{aligned}$$

2. Let $S := \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$.
   a) Find S^{-1}.
   b) For which constant(s) λ is the matrix $\begin{pmatrix} 2-\lambda & 5 \\ 1 & 3-\lambda \end{pmatrix}$ invertible?
   c) Let $D := \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ and $B := SDS^{-1}$. Compute B^{10}.

3. Solve all of the following equations. [Note that the left sides of these equations are identical.]

   a). $\begin{aligned} 2x + 5y &= \ \ 5 \\ x + 3y &= -1 \end{aligned}$ b). $\begin{aligned} 2x + 5y &= \ \ 0 \\ x + 3y &= -2 \end{aligned}$ c). $\begin{aligned} 2x + 5y &= 1 \\ x + 3y &= 0 \end{aligned}$ d). $\begin{aligned} 2x + 5y &= 2 \\ x + 3y &= 1 \end{aligned}$

4. Find all *invertible* $n \times n$ matrices A with the property $A^2 = 3A$.

5. Let A and B be 2×2 matrices.
   a) If B is invertible and $AB = 0$, show that $A = 0$.
   b) Give an example of (simple!) 2×2 matrices where $AB = 0$ but $BA \neq 0$.

Linear Equations as Maps The next two problems have an entirely different character. It is about viewing systems of equations as maps. Think of this an an introduction to *computer graphics*. We'll use these ideas throughout Math 312.

The standard technique goes back to Descartes' introduction of coordinates in geometry. Say one has two copies of the plane, the first with coordinates (x_1, x_2), the second with coordinates (y_1, y_2). Then the high school equations

$$x_1 - 0x_2 = y_1 \tag{1}$$

$$x_1 + \ \ x_2 = y_2 \tag{2}$$

can be thought of as a mapping from the (x_1, x_2) plane to the (y_1, y_2) plane. For instance, if $x_1 = 1$ and $x_2 = 0$, then $y_1 = 1$ and $y_2 = 1$. Thus the point $(1, 0)$ is mapped to the point $(1, 1)$.

6. Use the above map in the following:
   a) What are the images of $(0, 0)$, $(1, 0)$, $(1, \pi/4)$, and $(0, \pi/4)$?
   b) What is the image of the rectangle with vertices at $(0, 0)$, $(1, 0)$, $(1, \pi/4)$, and $(0, \pi/4)$? [Draw a sketch.]
   c) What is the image of the rectangle with vertices at $(1, 0)$, $(2, 0)$, and $(2, \pi/4)$, and $(1, \pi/4)$?

7. Next consider the *nonlinear* map from the (x_1, x_2) plane to the (y_1, y_2) plane

$$\begin{aligned} x_1 \cos x_2 &= y_1 \\ x_1 \sin x_2 &= y_2 \end{aligned}$$

   a) What are the images of $(0, 0)$, $(1, 0)$, $(1, \pi/4)$, and $(0, \pi/4)$?
   b) What is the image of the rectangle with vertices at $(0, 0)$, $(1, 0)$, $(1, \pi/4)$, and $(0, \pi/4)$? [Draw a sketch.]
   c) What is the image of the rectangle with vertices at $(1, 0)$, $(2, 0)$, and $(2, \pi/4)$, and $(1, \pi/4)$?

8. Later in the semester we will apply linear algebra to solve systems of linear ordinary differential equations. This will assume you know how to solve some sinple O.D.E.'s. As rust remover, solve the following differential equations for $u(t)$.
   a) $u'(t) = -2u$ with $u(0) = -3$.
   b) $u'(t) = -2tu$ with $u(0) = -2$.
   c) $u''(t) + 4u = 0$ with $u(0) = 0$ and $u'(0) = 1$.
   d) $u''(t) - 4u = 0$ with $u(0) = 0$ and $u'(0) = 1$.

9. Show that *every* solution of $u'' + 2u' + 2u = 0$ has the property that $\lim_{t\to\infty} u(t) = 0$.

[Last revised: January 1, 2013]