ODE-Matrix

Particular solution of $u''-3u = x^2$ using matrices. Let $\mathcal{P}_2$ be the linear space of polynomials of degree at most 2.

$$p(x) = a + bx + cx^2$$

Note that $L : \mathcal{P}_2 \to \mathcal{P}_2$. We want to represent L as a 3×3 matrix using the standard basis for $\mathcal{P}_2$, which we indicate by $\mathcal{B}$

$$p_0(x) = 1 = \begin{pmatrix}1\\0\\0\end{pmatrix}_{\mathcal{B}}, \quad p_1(x) = x = \begin{pmatrix}0\\1\\0\end{pmatrix}_{\mathcal{B}}, \quad p_2(x) = x^2 = \begin{pmatrix}0\\0\\1\end{pmatrix}_{\mathcal{B}}.$$

Then

$$Lp_0 = L1 = -3 = -3p_0 = \begin{pmatrix}-3\\0\\0\end{pmatrix}_{\mathcal{B}},$$

$$Lp_1 = Lx = -3x = -3p_1 = \begin{pmatrix}0\\-3\\0\end{pmatrix}_{\mathcal{B}},$$

$$Lp_2 = 2 - 3x_2 = 2p_0 - 3p_2 = \begin{pmatrix}2\\0\\-3\end{pmatrix}_{\mathcal{B}}.$$

In this basis, the matrix representing L has Lp_0 as its first column, Lp_1 as its second column, and Lp_2 as its third column.

$$L_{\mathcal{B}} = \begin{pmatrix}-3 & 0 & 2\\ 0 & -3 & 0\\ 0 & 0 & -3\end{pmatrix}_{\mathcal{B}}.$$

To solve $u'' - 3u = x^2$ we solve the matrix equation

$$\begin{pmatrix}-3 & 0 & 2\\ 0 & -3 & 0\\ 0 & 0 & -3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{1}$$

Of course this is exactly the same as seeking a solution of $u'' - 3u = x^2$ by seeking u as $u(x) = a + bx + cx^2$ since $Lu = 2c - 3(a + bx + cx^2)$ so we want to pick the unknown coefficients a, b, c so that

$$2c - 3(a + bx + cx^2) = x^2, \quad \text{that is,} \quad -3cx^2 - 3bx + (2c - 3a) = x^2$$

so

$$-3c = 1, \qquad -3b = 0, \qquad 2c - 3a = 0,$$

which is the same as (1)

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