Properties of Determinants

Let A be an $n \times n$ matrix with columns A_1, $A_2, \ldots, A_n$. Below are the properties of the *determinant* of A. We will often write them in terms of the columns of A: $\det A = \det(A_1, A_2, \ldots, A_n)$. It is often helpful to think of the determinant as the **volume** of the parallelepiped spanned by the columns. From this view, since the volume is the standard unit cube is 1 it is clear that we should want $\det I = 1$ and $\det(cI) = c^n$.

Note that except for small matrices (2×2 or 3×3) or very special matrices, determinants are difficult to compute. They are primarily a theoretical tool.

1. *The determinant is linear in each column.* For instance, if the second column A_2 is replaced by $A_2 + B_2$ and the other columns are unchanged, then

$$\det(A_1, A_2+B_2, A_3, \ldots, A_n) = \det(A_1, A_2, A_3, \ldots, A_n) + \det(A_1, B_2, A_3, \ldots, A_n) \qquad (1)$$

also, if one of the columns if multiplied by a scalar c and the other columns are unchanged, then the determinant is multiplied by c; for instance

$$\det(A_1, cA_2, A_3, \ldots, A_n) = c\det(A_1, A_2, \ldots, A_n). \qquad (2)$$

An immediate consequence is $\det(cA) = c^n \det A$

2. *If two columns are interchanged, the sign of the determinant is reversed.* For instance

$$\det(A_1, A_3, A_2, A_4, \ldots, A_n) = -\det(A_1, A_2, A_3, A_4, \ldots, A_n)$$

Consequence: *if two columns are the same, then the determinant is zero.* More generally, if two columns are linearly dependent, then the determinant is zero. Interpreting the determinant as volume, this is clear since then the column vectors of A span at most $n-1$ dimensions so the volume of the parallelepiped obtained using the columns of A is zero.

3. $\det I = 1$. In terms of volume, this says the determinant of the standard unit "cube" is 1, as mentioned above.

4. $\det A^T = \det A$, so it is equivalent to think of the properties of the determinant in terms of either the columns or rows.

5. If A and B are both $n \times n$ matrices, then $\det(AB) = (\det A)(\det B)$.
Consequence 1: if A is invertible, then

$$1 = \det I = \det(A^{-1}A) = (\det A^{-1})(\det A) \qquad \text{so} \qquad \det A^{-1} = \frac{1}{\det A}.$$

Consequence 2: If $A = S^{-1}BS$ (so A and B are *similar*, then $\det A = \det B$¿
Consequence 3: an *orthogonal matrix* R has the property that $R^{-1} = R^T$. Thus we see that

$$1 = \det I = \det(R^T R) = (\det R^T)(\det R) = (\det R)^2 \qquad \text{so} \qquad \det R = \pm 1.$$

A reflection, such as $R = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ has $\det R = -1$. For any linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$, computer graphics workers use that if $\det A < 0$, then A includes a reflection.

It turns out that all of the properties of determinants are consequences of properties 1–3. One example is that *If a multiple of one column is added to another column, the determinant is unchanged.* An example shows this. Say we add cA_3 to A_2. Then

$$\begin{aligned}\det(A_1, A_2 + cA_3, A_3, \dots, A_n) &= \det(A_1, A_2, A_3, \dots, A_n) + c\det(A_1, A_3, A_3, \dots, A_n)\\ &= \det(A_1, A_2, A_3, \dots, A_n) + 0,\end{aligned}$$

since the second and third columns in $\det(A_1, A_3, A_3, \dots, A_n)$ are identical.
Using this, it is easy to see that *the determinant of an upper triangular matrix is the product of the diagonal elements*. This also implies the important fact that *a matrix A is invertible if (and only if)* $\det A \neq 0$. We will prove this below.

One useful fact is that if a matrix A is invertible, one can use determinants to *explicitly* write a formula for the solution of the linear equations $AX = Y$ and thus get a formula for A^{-1}. We will do this in a moment. Because determinants are difficult to compute, this formula is less useful than one might anticipate.
If $X = (x_1, x_2, \dots, x_n)$ and, as above, $A_1, \dots, A_n$ are the column of A, then the equation $AX = Y$ means

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n = Y.$$

In the formula $\det A = \det(A_1, A_2, A_3, \dots, A_n)$, replace the first column by the above formula for Y and use equations (1) and (2) to find

$$\begin{aligned}\det(Y, A_2, A_3, \dots, A_n) &= \det(x_1A_1 + \sum_{j=2}^{n} x_jA_j,\ A_2, A_3, \dots, A_n)\\ &= \det(x_1A_1, A_2, A_3, \dots, A_n) + \det(\sum_{j=2}^{n} x_jA_j,\ A_2, A_3, \dots, A_n)\\ &= x_1 \det(A_1, A_2, A_3, \dots, A_n) + \sum_{j=2}^{n} x_j \det(A_j,\ A_2, A_3, \dots, A_n)\\ &= x_1 \det(A_1, A_2, A_3, \dots, A_n),\end{aligned}$$

where we observed that $\det(A_j,\ A_2, A_3, \dots, A_n) = 0$ for each $j \geq 2$ because the column A_j occurs in both the first slot and the j^{th} slot.
Solve the above equation for x_1 to find

$$x_1 = \frac{\det(Y, A_2, A_3, \dots, A_n)}{\det(A_1, A_2, A_3, \dots, A_n)}.$$

Note that the denominator is just $\det A$. Similarly,

$$x_2 = \frac{\det(A_1, Y, A_3, \dots, A_n)}{\det(A_1, A_2, A_3, \dots, A_n)}, \quad \dots, \quad x_n = \frac{\det(A_1, A_2, A_3, \dots, A_{n-1}, Y)}{\det(A_1, A_2, A_3, \dots, A_n)}.$$

This formula is called *Cramer's Rule*. It shows that if $\det A \neq 0$, then A is invertible. A tiny application is that if A is a matrix with integer elements and $\det A = \pm 1$, then the elements of A^{-1} are also integers.

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