Midterm Exam 2

Math 312, Fall 2012

You have 54 minutes.

No books, phones, calculators, talking, etc. You are allowed your own hand-written 3×5 inch note card, both sides, but **no other materials**. You may write on the back side of the sheets if necessary.

PLEASE WRITE LEGIBLY AND SHOW YOUR WORK.
YOU MAY NOT RECEIVE FULL CREDIT IF YOU DO NOT SHOW WORK.

Good luck!

Name ________________________________

Total Score ____________ (/48 points)

1. *(6 points)* Let $A = \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}$.

   (a) Diagonalize A. (Don't forget to find an inverse. Also, remember you can check your answer.)

   (b) Let $\mathcal{B}$ be the eigenbasis you found above. What is $[T]_{\mathcal{B}}$, where $T(\vec{x}) = A\vec{x}$?

2. *(6 points)* Compute the determinant of the following matrix. (Hint: the fastest method here does not use the long, co-factor expansion formula.)

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 2 & -4 & -1 \\ -2 & -6 & 2 & 3 \\ 3 & 7 & -3 & 8 \end{bmatrix}.$$

3. *(4 points)* Suppose a 4×4 matrix B has rank equal to 1 and trace equal to 10. What are eigenvalues of B? Be sure to describe any multiplicities, and explain your answer.

4. *(6 points)* Let

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}.$$

   (a) Find the eigenvalues of A.

   (b) Consider the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$. Which of the following describes how the system behaves for a given nonzero initial state $\vec{x}(0)$?

   (A) $\vec{x}(t)$ spirals in toward the origin as $t \to \infty$.
   (B) $\vec{x}(t)$ spirals out to infinity as $t \to \infty$.
   (C) $\vec{x}(t)$ oscillates periodically in an ellipse.
   (D) $\vec{x}(t)$ is repelled out to infinity as $t \to \infty$, but does not spiral.

5. *(8 points)* Let W be the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix}.$$

(a) Find an *orthonormal* basis of W.

(b) Find the orthogonal projection of

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

onto W.

6. *(10 points)* An $n \times n$ matrix is called *nilpotent* if A^k equals the zero matrix for some positive integer k. (For instance, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent.)

(a) Show that if λ is an eigenvalue of a nilpotent matrix A, then $\lambda = 0$. (Hint: start with the equation $A\vec{x} = \lambda\vec{x}$.)

(b) Show that if A is both nilpotent and diagonalizable, then A is the zero matrix. (Hint: use (a).)

(c) Let A be the matrix that represents $T : \mathbb{P}_5 \to \mathbb{P}_5$ given by differentiation. Without doing any computations, explain why A must be nilpotent.

7. *(8 points)* Let $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be given by $T(f) = f''''$ (i.e., T sends a function to its fourth derivative).

   (a) Find a basis for the 0-eigenspace.

   (b) Find a basis for the 1-eigenspace. (Hint: it's four-dimensional.)

8. *(3 points BONUS)* Must a 3×3 real matrix with a complex (non-real) eigenvalue be diagonalizable? Explain.