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Math 312 **Exam 2** Jerry L. Kazdan

March 21, 2013 9:00 – 10:20

DIRECTIONS This exam has two parts. Part A has shorter 5 questions, (10 points each so total 50 points) while Part B had 5 problems (15 points each, so total is 75 points). Maximum score is 125 points.

Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

PART A: Five short answer questions (10 points each, so 50 points).

A–1. Let A be a 5×5 real matrix with $\det A = -1$. What is $\det(-2A)$?

A–2. We consider the equation $A\mathbf{x} = \mathbf{b}$ where $\mathbf{x}$ and $\mathbf{b}$ are in $\mathbb{R}^4$ and A is a 4×4 matrix with determinant 7. True or False – and Why?

a) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

b) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

c) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has no solution.

d) For all vectors $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

A–3. A matrix is *nilpotent* if $A^k = 0$ for some positive integer k [Example: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$]

If λ is an eigenvalue of a nilpotent matrix, show that $\lambda = 0$ [SUGGESTION: Begin with $A\vec{v} = \lambda\vec{v}$].

A–4. In $\mathbb{R}^4$, find the distance from the point $(1, -2, 0, 3)$ to the "plane" $x_1 + 3x_2 - x_3 + x_4 = 0$.

A–5. In $\mathbb{R}^n$ with the usual inner product, show that

$$\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2 = 4\langle \vec{x},\, \vec{y}\rangle$$

PART B Five questions, 15 points each (so 75 points total).

B–1. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$.

B–2. The matrix $B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Use this to solve the differential equation $\dfrac{d\vec{x}(t)}{dt} = B\vec{x}(t)$ with initial condition $\vec{x}(0) = (2,\, 0)$.

B–3. For certain polynomials **f**, **g**, and **h** say we are given the following table of inner products:

$\langle\ ,\ \rangle$	**f**	**g**	**h**
f	4	0	8
g	0	1	0
h	8	0	50

For example, $\langle \mathbf{g},\, \mathbf{h}\rangle = \langle \mathbf{h},\, \mathbf{g}\rangle = 0$. Let E be the span of **f** and **g**.

a) Compute $\langle \mathbf{f},\, \mathbf{g} + \mathbf{h}\rangle$.

b) Compute $\|\mathbf{g} + \mathbf{h}\|$.

c) Find $\mathrm{Proj}_E \mathbf{h}$. [Express your solution as linear combinations of **f** and **g**.]

d) Find an orthonormal basis of the span of **f**, **g**, and **h**. [Express your results as linear combinations of **f**, **g**, and **h**.]

B–4. Consider the space $\mathcal{P}_2$ of polynomials of degree at most two with the following inner product:

$$\langle p,\, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

a) Compute the inner product of the polynomials $p(x) := 1$ and $q(x) := x$.

b) Using this inner product, find an orthogonal basis for the space $\mathcal{P}_2$.

B–5. Say you have done an experiment and obtained the data points $(-1, 1)$, $(0, -1)$, $(1, -1)$, and $(2, 3)$. Based on some other evidence you believe this data should fit a curve of the form $y = a + bx^2$.

a) Write the (over-determined) system of linear equations you would ideally like to solve for the unknown coefficients a and b.

b) Use the method of least squares to find the *normal equations* for the coefficients a and b.

c) Solve the normal equations to find the coefficients a and b *explicitly* (numbers, like 3/5 and -2).