Math 312, Fall 2012 Jerry L. Kazdan

Problem Set 9

DUE: In class Tuesday, Nov. 27 *Late papers will be accepted until 12:00 on Thursday (at the beginning of class).*

1. Suppose that λ is an eigenvalue of an $n \times n$ matrix A and let E_λ be the set of all eigenvectors with the *same* eigenvalue λ. Show that E_λ is a linear subspace of $\mathbb{R}^n$.

   SOLUTION: Say that both $\vec{v}$ and $\vec{w}$ are in E_λ. We need to show that both $c\vec{v}$ and $\vec{v}+\vec{w}$ are in E_λ cor any scalar c.

   Now $A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda(c\vec{v}$ so $c\vec{v} \in E_\lambda$.

   Also, $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w} = \lambda\vec{v} + \lambda\vec{w} = \lambda(\vec{v}+\vec{w})$. Thus $\vec{v}+\vec{w} \in E_\lambda$.

2. Let A be a 2×2 real matrix whose eigenvalues are not real.

   a) Suppose one of the eigenvalues has absolute value 1. Explain why the other must as well.

      SOLUTION: Let $\lambda = \alpha + i\beta$ be a complex eigenvalue with $\beta \neq 0$. Since A is real, then its complex conjugate, $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue. But $|\bar{\lambda}| = |\lambda| = 1$.

   b) Explain why A must be diagonalizable.

      SOLUTION: Since the eigenvalues of A are distinct, it is diagonalizable.

3. This asks you to come up with four examples. In each case, find a matrix (perhaps 2×2) that is:

   a) Both invertible and diagonalizable.

      SOLUTION: The identity matrix, I; the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

   b) Not invertible, but diagonalizable.

      SOLUTION: The zero matrix, 0; the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$.

   c) Not diagonalizable but is invertible.

      SOLUTION: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

   d) Neither invertible nor diagonalizable.

      SOLUTION: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

4. If the matrices A and B are similar and if $A^3 = 0$, must $B^3 = 0$? Proof or counterexample.

   SOLUTION: True. Since $B = S^{-1}AS$ for some S, then $B^3 = S^{-1}A^3S = 0$.

5. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs.

   One has data on which location the cars are returned daily:

   - RENTED AT AIRPORT: 2% are returned to the City and 25% to the Suburbs. The rest are returned to the Airport.
   - RENTED IN CITY : 10% returned to Airport, 10% returned to Suburbs.
   - RENTED IN SUBURBS: 25% are returned to the Airport and 2% to the city.

   If initially there are 35 cars at the Airport, 150 in the city, and 35 in the suburbs, what is the long-term distribution of the cars?

   SOLUTION: Lat A_k, C_k, and S_k be the number of cars on day k at the Airport, City, and Suburbs, respectively. Then

$$\begin{aligned} A_{k+1} =& .73A_k + .10C_k + .25S_k \\ C_{k+1} =& .02A_k + .80C_k + .02S_k \\ S_{k+1} =& .25A_k + .10C_k + .73S_k \end{aligned}$$

   Thus the transition matrix T is: $T = \begin{pmatrix} .73 & .10 & .25 \\ .02 & .80 & .02 \\ .25 & .10 & .73 \end{pmatrix}.$

   To find the he eigenvector P with eigenvalue 1 one needs to solve:

$$\begin{aligned} -27A + 10C + 25S =& 0 \\ 2A - 20C + \ \ 2S =& 0 \\ 25A + 10C - 27S =& 0 \end{aligned}$$

   This gives $A = S = 5C$.

   In addition, since P is supposed to be a probability vector, $A + C + S = 1$. Thus $C = 1/11$so $A = S = 5/11$.

   Using the initial state, there are $35 + 150 + 35 = 220$ cars in all. Thus in the long run:
   city: 220/11=20 cars
   airport: 100 cars
   suburbs: 100 card.

6. Let R be a (real) 3×3 orthogonal matrix.

   a) Show that the eigenvalues, λ, which may be complex, all have absolute value 1.

      SOLUTION: Since R is an orthogonal matrix, then $\|R\vec{v}\| = \|\vec{v}\|$ for every vector $\vec{v}$. In particular, if $R\vec{v} = \lambda\vec{v}$, then $\|\vec{v}\| = \|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$ so $|\lambda| = 1$.

b) If $\det R = 1$ show that $\lambda = 1$ is one of the eigenvalues of R and that if $R \neq I$, no other eigenvalue can be 1.

SOLUTION: Since R is a real matrix, either two of its eigenvalues are complex or none are.

CASE 1 Two complex eigenvalues, say λ_1 and λ_2, so $\lambda_2 = \bar{\lambda_1}$ and then $\lambda_1\lambda_2 = 1$. Since $1 = \det R = \lambda_1\lambda_2\lambda_3$, then $\lambda_3 = 1$.

CASE 2 All three eigenvalues are real. Since the real eigenvalues can only have values ± 1, the only possibilities are

$$1,\ 1,\ 1, \qquad 1,\ 1,\ -1, \qquad 1,\ -1,\ -1, \qquad -1,\ -1,\ -1.$$

Because $\det R = 1$ and $R \neq I$, the only possibility is $1,\ -1,\ -1$.

For the remainder of this problem assume $\det R = 1$ *and* $R \neq I$.

c) Let N be an eigenvector corresponding to $\lambda = 1$ and . let Q be the plane of all vectors orthogonal to N. Show that R maps Q to Q.

SOLUTION: An orthogonal matrix preserves the inner product: $\langle R\vec{x},\, \mathbb{R}\vec{y}\rangle = \langle \vec{x},\, \vec{y}\rangle$ for all vectors $\vec{x}$, $\vec{y}$. Thus, since $R\vec{N} = \vec{N}$, if $\vec{x} \in Q$, so $\langle \vec{x},\, \vec{N}\rangle = 0$, then

$$\langle R\vec{x},\, \vec{N}\rangle = \langle R\vec{x},\, R\vec{N}\rangle = \langle \vec{x},\, \vec{N}\rangle = 0,$$

that is, $R\vec{x} \in Q$.

d) Why does this show that R is a rotation of the plane Q with N as the axis of rotation?

SOLUTION: Q is a two dimensional plane and R is an orthogonal transformation from Q to itself. Thus R acts on Q as either a rotation or a reflection. Since $\det R = 1$, it is a rotation.

7. [BRETSCHER, SEC. 7.1 #36] Find a 2×2 matrix A such that $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are eigenvectors with corresponding eigenvalues 5 and 10.

SOLUTION: There are several approaches. Here is one. Since A can be diagonalized (it has a basis of eigenvectors) the matrix $S = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ whose columns are the eigenvectors of A has the property that $S^{-1}AS = D$, where D is the diagonal matrix whose elements are 5 and 10. Thus

$$A = SDS^{-1} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -2 & 11 \end{pmatrix}.$$

8. [Bretscher, Sec. 7.1 #38] We are told that $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix $M := \begin{pmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{pmatrix}$. What is the associated eigenvalue?

Solution:

$$\begin{pmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

so the eigenvalue is 2.

9. [Bretscher, Sec. 7.2 #12] Find all of the eigenvalues of $M := \begin{pmatrix} 2 & -2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 2 & -3 \end{pmatrix}$ and determine their algebraic multiplicity.

Solution:

$$\begin{aligned} \det(M - \lambda I) = &\det \begin{pmatrix} 2-\lambda & -2 \\ 1 & -1-\lambda \end{pmatrix} \det \begin{pmatrix} 3-\lambda & -4 \\ 2 & -3-\lambda \end{pmatrix} \\ =&(\lambda^2 - \lambda)(\lambda^2 - 1). \end{aligned}$$

The eigenvalues are thus $0,\ 1,\ 1,\ -1$. The eigenvalue 1 has algebraic multiplicity 2, the others have algebraic multiplicity 1.

For this example the geometric multiplicities agree with the algebraic multiplicities so the matrix can be diagonalized.

10. [Bretscher, Sec. 7.2 #14] Consider a 4×4 matrix $A := \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where B, C, and D are 2×2 matrices. What is the relationship between the eigenvalues of A, B, C, and D?

Solution: First a preliminary result. In the above notation, if $M := \begin{pmatrix} R & S \\ 0 & T \end{pmatrix}$, then $\det M = (\det R)(\det T)$.

Case 1: If R is not invertible then its columns – and hence the corresponding columns of M – are linearly dependent so both $\det R = 0$ and $\det M = 0$.

Case 2: R is invertible. This is an exercise in block multiplication of matrices. Let X be an unknown 2×2 matrix. Then

$$\begin{pmatrix} R & S \\ 0 & T \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} R & RX + S \\ 0 & T \end{pmatrix}.$$

Since R is invertible we can pick X so that $RX + S = 0$. The result should now be clear.

Applying this to $M := A = \lambda I$ we conclude that the eigenvalues of A, including their algebraic multiplicities, are exactly those determined by B and D. Note, however, that even if both B and D are diagonalizable, A may not be. The simplest example is $B = D = 0$ and $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

11. The characteristic polynomial $p_A(\lambda)$ of A is

$$p_A(\lambda) := \det(A - \lambda I) = (-\lambda)^n + c_{n-1}(-\lambda)^{n-1} + \cdots + c_0, \tag{1}$$

Caution: many books define the characteristic polynomial as $\det(\lambda I - A)$, which changes some signs.) In class we showed that similar matrices have the *same* characteristic polynomial.

Recall that the *trace* of a matrix $A = (a_{ij})$ is the sum of the diagonal elements: $\operatorname{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$. In this problem you will see that the trace and determinant of A are two of the coefficients in the characteristic polynomial.

a) Show that $c_0 = \det(A)$.

SOLUTION: Let $\lambda = 0$ in equation (1).

b) Since also the eigenvalues of A are the roots of the characteristic polynomial, show that the trace of A is the sum of its eigenvalues:

$$\operatorname{trace}(A) = \lambda_1 + \cdots + \lambda_n.$$

These are the coefficient c_{n-1} of $(-\lambda)^{n-1}$. [Although this is true for all n., only do this for $n = 3$. The procedure in the general case is identical.]

SOLUTION: In the case of $n = 3$, the characteristic polynomial is a cubic. We compute the coefficient of λ^2 for both sides of equation (1). First we expand $\det(\lambda I - A)$ by minors using the first column:

$$\begin{aligned} p_A(\lambda) &= \det \begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix} \\ &= (a_{11} - \lambda) \det \begin{pmatrix} a_{22} - \lambda & a_{23} \\ a_{32} & a_{33} - \lambda \end{pmatrix} - a_{21} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} - \lambda \end{pmatrix} + a_{31} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} - \lambda \end{pmatrix}. \end{aligned}$$

Note that the second and third terms on the last line do not contribute any quadratic terms in λ so we can ignore them. We indicate such terms by writing $+\cdots$. Continuing from above

$$\begin{aligned} p_A(\lambda) &= (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) + \cdots \\ &= -\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 + \cdots \end{aligned} \tag{2}$$

Next we factor $p_A(\lambda)$ using its roots, the eigenvalues λ_j. The computation is similar to that above.

$$p_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) = -\lambda^3 + (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + \cdots . \qquad (3)$$

Comparing the coefficient of λ^2 in equations (2) and (3) we conclude that the trace of a matrix is the sum of its eigenvalues.

12. Let A be the transition matrix of a Markov Chain. If $\vec{v} := (1, 1. \ldots, 1)^T$ show that $A^*\vec{v} = \vec{v}$.

    Why does this imply that $\lambda = 1$ is an eigenvalue of A? Thus if A is the transition matrix of any Markov Chain, $\lambda = 1$ is always an eigenvalue.

    SOLUTION: Since the sum of the elements in each column of A is 1, the sum of the elements in each row of A^* is 1. This is exactly the same as $A^*\vec{v} = \vec{v}$.

    Because for any matrix A the eigenvalues of A and A^* are the same, we conclude that 1 is an eigenvalue of A.

13. Say a sequence $x = \{x_0, x_1, x_2, x_3, \ldots\}$ has the properties $x_0 = 0$, $x_1 = 1$, and, recursively, $x_{k+2} = x_{k+1} + x_k$ for $k = 0, 1, 2, 3 \ldots$. For instance, $x_2 = 1$, $x_3 = 2$, etc. Let $u_k = x_k$, $v_k = x_{k+1}$ and write $W_k := \begin{pmatrix} u_k \\ v_k \end{pmatrix}$. Note $W_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

    a) Show that

    $$W_{k+1} = \begin{pmatrix} v_k \\ v_k + u_v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} W_k.$$

    SOLUTION: Since $v_{k+1} = x_{k+2} = x_k + x_{k+1} = u_k + v_k$, then

    $$W_{k+1} = \begin{pmatrix} u_{k+1} \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} v_k \\ u_k + v_k \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} W_k = AW_k,$$

    where A is the evident 2×2 matrix.

    b) Let A denote the 2×2 matrix above. Show that $W_k = A^k W_0$.

    SOLUTION: $W_2 = AW_1 = A^2 W_0$, $W_3 = AW_2 = A^3 W_0$, etc.

    c) Diagonalize A and use this to compute A^k and thus W_k explicitly.

    SOLUTION: The characteristic polynomial of A gives $\lambda^2 - \lambda - 1 = 0$, so $\lambda_\pm = \frac{1}{2}(1 \pm \sqrt{5}$. The corresponding eigenvectors are $\vec{v}_+ = \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix}$ and $\vec{v}_- = \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$. Then $S^{-1}AS = D$, where $D = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$, and the change of coordinates matrix is $S = \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}$ whose columns are the corresponding eigenvectors of A. Thus $A = SDS^{-1}$. Therefore, using $\lambda_+\lambda_- = -1$,

$$A^k = SD^kS^{-1} = \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} \lambda_+^k & 0 \\ 0 & \lambda_-^k \end{pmatrix} \frac{-1}{\sqrt{5}} \begin{pmatrix} \lambda_- & -1 \\ -\lambda_+ & 1 \end{pmatrix}$$
$$= \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_+^{k-1} - \lambda_-^{k-1} & \lambda_+^k - \lambda_-^k \\ \lambda_+^k - \lambda_-^k & \lambda_+^{k+1} - \lambda_-^{k+1} \end{pmatrix}.$$

Therefore

$$W_k = A^k W_0 = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_+^k - \lambda_-^k \\ \lambda_+^{k+1} - \lambda_-^{k+1} \end{pmatrix}.$$

d) Use this to get an explicit formula for x_k.

SOLUTION: Since $x_k = u_k$, then x_k is the first component of W_k,

$$x_k = \frac{1}{\sqrt{5}} \left(\lambda_+^k - \lambda_-^k \right) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right].$$

It is interesting that although by the recursive formula $x_{k+2} = x_k + x_{k+1}$ the x_k are clearly integers, the explicit formula we just found involves $\sqrt{5}$.

14. [BRETSCHER, SEC. 7.3 #28] Let $B := \begin{pmatrix} k & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & k \end{pmatrix}$ where k is an arbitrary constant. Find the eigenvalue(s) of B and determine both their algebraic and geometric multiplicities. [NOTE: First try the analogous 2×2 case.]

SOLUTION: Since B is upper triangular its eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = k$ (so the algebraic multiplicity is 4).

To determine the geometric multiplicity we want the dimension of the kernel of $B - kI$, that is, the solutions of

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Clearly v_1 can be anything, while $v_2 = v_3 = v_4 = 0$. Thus the kernel is just multiples of $\vec{v} = (1, 0, 0, 0)$ and is one dimensional. The geometric dimension of this eigenvalue is 1.

15. [BRETSCHER, SEC. 7.4 #21-22] Let $A = \begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}$ and $B := \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$. For which choices of the constants a and b are these diagonalizable?

SOLUTION: The matrix A has distinct eigenvalues 1 and 2 so it can be diagonalized for any choice of a.

Similarly for B, if $b \neq 1$, it can be diagonalized for any choice of a. However, if $b = 1$, then $\lambda = 1$ has algebraic multiplicity 2 but if $a \neq 0$ it has geometric multiplicity 1. It can be diagonalized if and only if $a = 0$.

REMARK I found the problems Bretscher, Sec. 7.3 #43-44 and many other applications at the end of Section 7.3 interesting. You might too. These are *not* assigned.

Bonus Problem

[Please give this directly to Professor Kazdan]

1-B Let $\mathcal{S}$ be the space of smooth real-valued functions $u(x)$ that periodic with period 2π, so $u(x+2\pi) = u(x)$ for all $x \in \mathbb{R}$ with the inner product $\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Let $D : \mathcal{S} :\to \int$ be the derivative operator, $Du = du/dx$ and let $L := -D^2$.

a) Find the adjoints of both D and L. Note that by definition, the adjoint A^* of a linear map A is the map that satisfies the identity $\langle v, Aw\rangle = \langle A^*v, w\rangle$ for all v and w. [HINT: Integrate by parts].

b) Find the eigenfunctions $u_k(x)$ and corresponding eigenvalues, λ_k of L, so $u(x)$ is 2π periodic and $Lu_k = \lambda_k u_k$.

c) Use the result of the previous part to conclude that if $k \neq \ell$ are integers, then $\sin kx$ is orthogonal to $\sin \ell x$ and $\cos \ell x$.

[Last revised: December 10, 2012]