Problem Set 9

DUE: In class Tuesday, Nov. 27 *Late papers will be accepted until 12:00 on Thursday (at the beginning of class).*

1. Suppose that λ is an eigenvalue of an $n \times n$ matrix A and let E_λ and let E_λ be the set of all eigenvectors with the *same* eigenvalue λ. Show that E_λ is a linear subspace of $\mathbb{R}^n$.

2. Let A be a 2×2 real matrix whose eigenvalues are not real.
   a) Suppose one of the eigenvalues has absolute value 1. Explain why the other must as well.
   b) Explain why A must be diagonalizable.

3. This asks you to come up with four examples. In each case, find a matrix (perhaps 2×2) that is:
   a) Both invertible and diagonalizable.
   b) Not invertible, but diagonalizable.
   c) Not diagonalizable but is invertible.
   d) Neither invertible nor diagonalizable.

4. If the matrices A and B are similar and if $A^3 = 0$, must $B^3 = 0$? Proof or counterexample.

5. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs.

   One has data on which location the cars are returned daily:

   - RENTED AT AIRPORT: 2% are returned to the City and 25% to the Suburbs. The rest are returned to the Airport.
   - RENTED IN CITY : 10% returned to Airport, 10% returned to Suburbs.
   - RENTED IN SUBURBS: 25% are returned to the Airport and 2% to the city.

   If initially there are 35 cars at the Airport, 150 in the city, and 35 in the suburbs, what is the long-term distribution of the cars?

6. Let R be a (real) 3×3 orthogonal matrix.
   a) Show that the eigenvalues, λ, which may be complex, all have absolute value 1.

b) If $\det R = 1$ show that $\lambda = 1$ is one of the eigenvalues of R and that if $R \neq I$, no other eigenvalue can be 1.

For the remainder of this problem assume $\det R = 1$ *and* $R \neq I$.

c) Let N be an eigenvector corresponding to $\lambda = 1$ and let Q be the plane of all vectors orthogonal to N. Show that R maps Q to Q.

d) Why does this show that R is a rotation of the plane Q with N as the axis of rotation?

7. [Bretscher, Sec. 7.1 #36] Find a 2×2 matrix A such that $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are eigenvectors with corresponding eigenvalues 5 and 10.

8. [Bretscher, Sec. 7.1 #38] We are told that $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix $\begin{pmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{pmatrix}$. What is the associated eigenvalue?

9. [Bretscher, Sec. 7.2 #12] Find all of the eigenvalues of $\begin{pmatrix} 2 & -2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 2 & -3 \end{pmatrix}$ and determine their algebraic multiplicity.

10. [Bretscher, Sec. 7.2 #14] Consider a 4×4 matrix $A := \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where B, C, and D are 2×2 matrices. What is the relationship between the eigenvalues of A, B, C, and D?

11. The characteristic polynomial $p_A(\lambda)$ of A is

$$p_A(\lambda) := \det(A - \lambda I) = (-\lambda)^n + c_{n-1}(-\lambda)^{n-1} + \cdots + c_0, \qquad (1)$$

Caution: many books define the characteristic polynomial as $\det(\lambda I - A)$, which changes some signs.) In class we showed that similar matrices have the *same* characteristic polynomial.

Recall that the *trace* of a matrix $A = (a_{ij})$ is the sum of the diagonal elements: $\operatorname{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$. In this problem you will see that the trace and determinant of A are two of the coefficients in the characteristic polynomial.

a) Show that $c_0 = \det(A)$.

b) Since also the eigenvalues of A are the roots of the characteristic polynomial, show that the trace of A is the sum of its eigenvalues:

$$\text{trace}(A) = \lambda_1 + \cdots + \lambda_n$$

These are the coefficient of $(-\lambda)^{n-1}$. [Although this is true for all n., only do this for $n = 3$. The procedure in the general case is identical.]

12. Let A be the transition matrix of a Markov Chain. If $\vec{v} := (1, 1 \ldots, 1)^T$ show that $A^*\vec{v} = \vec{v}$.

    Why does this imply that $\lambda = 1$ is an eigenvalue of A? Thus if A is the transition matrix of any Markov Chain, $\lambda = 1$ is always an eigenvalue.

13. Say a sequence $x = \{x_0, x_1, x_2, x_3, \ldots\}$ has the properties $x_0 = 0$, $x_1 = 1$, and, recursively, $x_{k+2} = x_{k+1} + x_k$ for $k = 0, 1, 2, 3 \ldots$. For instance, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$, etc. Let $u_k = x_k$, $v_k = x_{k+1}$ and write $W_k := \begin{pmatrix} u_k \\ v_k \end{pmatrix}$. Note $W_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

    a) Show that
    $$W_{k+1} = \begin{pmatrix} v_k \\ v_k + u_v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} W_k.$$

    b) Let A denote the 2×2 matrix above. Show that $W_{k+1} = A^k W_1$.

    c) Diagonalize A and use this to compute A^k and thus W_k explicitly.

    d) Use this to get an explicit formula for x_k.

14. [Bretscher, Sec. 7.3 #28] Let $B := \begin{pmatrix} k & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & k \end{pmatrix}$ where k is an arbitrary constant. Find the eigenvalue(s) of B and determine both their algebraic and geometric multiplicities. [Note: First try the analogous 2×2 case.]

15. [Bretscher, Sec. 7.4 #21-22] Let $A = \begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}$ and $B := \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$. For which choices of the constants a and b are these diagonalizable?

Remark I found the problems Bretscher, Sec. 7.3 #43-44 and many other applications at the end of Section 7.3 interesting. You might too. These are *not* assigned.

Bonus Problem

[Please give this directly to Professor Kazdan]

1-B Let $\mathcal{S}$ be the space of smooth real-valued functions $u(x)$ that periodic with period 2π, so $u(x+2\pi) = u(x)$ for all $x \in \mathbb{R}$ with the inner product $\langle f,\, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Let $D : \mathcal{S} :\to \int$ be the derivative operator, $Du = du/dx$ and let $L := -D^2$.

a) Find the adjoints of both D and L. Note that by definition, the adjoint A^* of a linear map A is the map that satisfies the identity $\langle v,\, Aw\rangle = \langle A^*v,\, w\rangle$ for all v and w. [HINT: Integrate by parts].

b) Find the eigenfunctions $u_k(x)$ and corresponding eigenvalues, λ_k of L, so $u(x)$ is 2π periodic and $Lu_k = \lambda_k u_k$.

c) Use the result of the previous part to conclude that if $k \neq \ell$ are integers, then $\sin kx$ is orthogonal to $\sin \ell x$ and $\cos \ell x$.

[Last revised: December 10, 2012]