Problem Set 8

DUE: In class Thursday, Nov. 8 *Late papers will be accepted until 1:00 PM Friday.*

Some of this is on the material in Bretscher, Sec. 5.5, concerning inner products in spaces of functions. *No* new ideas are involved, but it does take time to simply relax.

1. For a square matrix A, a scalar λ is an *eigenvalue* and a vector $\vec{v} \neq 0$ is a corresponding *eigenvector* if $A\vec{v} = \lambda\vec{v}$, so A maps $\vec{v}$ to a multiple of itself.

   If A is a symmetric (that is, self-adjoint) matrix with eigenvalues λ, μ, $\lambda \neq \mu$ and corresponding eigenvectors $\vec{v}$ and $\vec{w}$. Show that $\vec{v}$ and $\vec{w}$ are orthogonal.

   SOLUTION: We know that $A\vec{v} = \lambda\vec{v}$ and $A\vec{w} = \mu\vec{w}$, where $\mu \neq \lambda$. Using the inner product we have

   $$\langle A\vec{v},\, \vec{w}\rangle = \lambda\langle \vec{v},\, \vec{w}\rangle \qquad \text{and} \qquad \langle A\vec{w},\, \vec{v}\rangle = \mu\langle \vec{w},\, \vec{v}\rangle.$$

   But since $A = A^*$, then $\langle A\vec{v},\, \vec{w}\rangle = \langle \vec{v},\, A\vec{w}\rangle\langle A\vec{w},\, \vec{v}\rangle$. Consequently,

   $$\lambda\langle \vec{v},\, \vec{w}\rangle = \mu\langle \vec{v},\, \vec{w}\rangle.$$

   Since $\lambda\nu\mu$, then $\langle \vec{v},\, \vec{w}\rangle - = 0$.

2. Introduce the following inner product on the space of continuous functions on the interval $-1 \leq x \leq 1$: $\langle f,\, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

   a) Show that $1 \perp x$.

   SOLUTION: $\langle 1,\, x\rangle = \int_{-1}^{1} 1 \cdot x\,dx = \frac{1}{2}x^2 \Big|_{-1}^{1} = 0$.

   b) For which constants a, b is $f(x) := a + bx + x^2$ orthogonal to both 1 and x?

   SOLUTION: We want $\langle a + bx + x^2,\, 1\rangle = 0$ and $\langle a + bx + x^2,\, x\rangle = 0$, that is

   $$0 = \int_{-1}^{1} (a + bx + x^2)1\,dx = 2a + 1 \qquad \text{and} \qquad 0 = \int_{-1}^{1} (a + bx + x^2)x\,dx = b,$$

   so $a = -1/2$ and $b = 0$.

   c) Find an orthogonal basis for the span of 1, x, and x^2.

   SOLUTION: $e_1(x) = 1$, $e_2(x) = x$, $e_3(x) = -\frac{1}{2} + x^2$.

3. A real-valued function is called *even* if $f(-x) = f(x)$ for all x, and *odd* if $f(-x) = -f(x)$ for all x. For instance, $2x^4 + x\sin 3x$ is even and $\sin 4x - 7x^5$ is odd. . Using the same inner product as above,

a) Show that any odd function $f(x)$ is orthogonal to the function 1.

SOLUTION: Since $f(x)$ is odd, using the substitution $t = x$ in the second step

$$\begin{aligned}\langle f,\, 1\rangle = \int_{-1}^{1} f(x)\cdot 1\, dx &= \int_{-1}^{0} f(x)\, dx + \int_{0}^{1} f(x)\, dx \\ &= \int_{0}^{1} f(-t)\, dt + \int_{0}^{1} f(x)\, dx \\ &= -\int_{0}^{1} f(t)\, dt + \int_{0}^{1} f(x)\, dx = 0.\end{aligned}$$

b) Show that any even function is orthogonal to $\sin 13x$.

SOLUTION: Almost identical to part (a).

c) Show that the product of an even function $f(x)$ and an odd function $g(x)$is odd.

SOLUTION: Let $h(x) = f(x)g(x)$. Then

$$h(-x) = f(-x)g(-x) = -f(x)g(x) = -h(x).$$

d) Show that any even function $f(x)$ is orthogonal to any odd function $g(x)$.

SOLUTION: Let $h(x) = f(x)g(x)$. Since h is odd, this follows from part (a).

4. [BRETSCHER, SEC. 5.5 #16] Consider the space of continuous functions on the interval $[0,\, 1]$ (that is, $0 \le x \le 1$) with the inner product $\langle f,\, g\rangle := \int_0^1 f(x)g(x)\, dx$.

a) Using this inner product, find an orthonormal basis for the space $\mathcal{P}_1$ of polynomials of degree at most one.

SOLUTION: We use the Gram-Schmidt process. Pick the constant c so that $x = x \cdot 1 + w$, where $w(x) \perp 1$. Then

$$\langle x,\, 1\rangle = c\langle 1,\, 1\rangle + \langle w,\, 1\rangle = c\int_0^1 1^2\, dx = c + 0 = c.$$

Since $\langle x,\, 1\rangle = \int_0^1 x\, dx = 1/2$, then $c = 1/2$. Consequently $w(x) = x - \frac{1}{2}$ is orthogonal to the function 1. Therefore $v_1(x) = 1$ and $v_2(x) = x - \frac{1}{2}$ is an orthogonal basis.

Because $\|v_1\|^2 = \int_0^1 1^2\, dx = 1$ and $\|v_2\|^2 = \int_0^1 (x - \frac{1}{2})^2\, dx = \frac{1}{12}$, an orthonormal basis is $e_1(x) = 1$ and $e_2(x) = \sqrt{12}(x - \frac{1}{2})$.

b) Find a linear polynomial $g(x) = a + bx$ that best approximates x^2 in the norm defined by this inner product.

SOLUTION: The best approximation (in this norm) of $f(x) = x^2$ by a funcrion of the form $a + bx$ is the orthogonal projection of x^2 into this space. We thus seek constants α and β so that

$$x^2 = \alpha e_1(x) + \beta e_2(x) + w(x), \tag{1}$$

where $w(x)$ is orthogonal to 1 and x, or equivalently, to e_1 and e_2. To find α and β, as usual we take the inner product of (1) with e_1 and e_2 to find

$$\alpha = \langle x^2, e_1 \rangle = \int_0^1 x^2 \cdot 1\, dx = \frac{1}{3} \quad \text{and} \quad \beta = \langle x^2, e_2 \rangle = \sqrt{12} \int_0^1 x^2 \left(x - \frac{1}{2} \right) = \frac{\sqrt{12}}{12} = \frac{1}{\sqrt{12}}.$$

Thus the best approximation (using this inner product) is $\frac{1}{3} + \left(x - \frac{1}{2}\right)$.

5. [Bretscher, Sec. 5.5 #20]. In $\mathbb{R}^2$ consider the inner product $\ll \vec{v}, \vec{w} \gg := \vec{v}^T \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} \vec{w}$ with corresponding norm $|||\vec{v}|||^2 := \ll \vec{v}, \vec{v} \gg$.

   a) Find all vectors in $\mathbb{R}^2$ that are orthogonal to $\vec{v} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

   Solution: The condition $\vec{w} := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ being orthogonal to $\vec{v}$ means

$$0 = \ll \vec{v}, \vec{w} \gg = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w_1 + 2w_2,$$

   so $\vec{w} = c \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ where c is any scalar.

   b) Find an orthonormal basis for $\mathbb{R}^2$ with respect to this inner product.

   Solution: The vectors $\vec{v}$ and $\vec{w}$ are an orthogonal basis. To get an orthonormal basis we just make them into unit vectors – using the norm associated with this new inner product $|||\vec{v}|||^2 := \ll \vec{v}, \vec{v} \gg = \vec{v}^T \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} \vec{v}$. Then $|||\vec{v}|||^2 = 1$ and $|||\vec{w}|||^2 = 4c^2$ so one orthonormal basis is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}.$$

6. [Bretscher, Sec. 5.5 #24]. Using the inner product of problem 4, for the polynomials $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ say we are given the following table of inner products:

$\langle\ ,\ \rangle$	**f**	**g**	**h**
f	4	0	8
g	0	1	3
h	8	3	50

For example, $\langle \mathbf{g}, \mathbf{h} \rangle = \langle \mathbf{h}, \mathbf{g} \rangle = 3$. Let E be the span of $\mathbf{f}$ and $\mathbf{g}$.

a) Compute $\langle \mathbf{f},\, \mathbf{g}+\mathbf{h}\rangle$.

SOLUTION: $\langle \mathbf{f},\, \mathbf{g}+\mathbf{h}\rangle = 0+8 = 8$.

b) Compute $\|\mathbf{g}+\mathbf{h}\|$.

SOLUTION: $\|\mathbf{g}+\mathbf{h}\|^2 = 1+2\cdot 3+50 = 57$ so $\|\mathbf{g}+\mathbf{h}\| = \sqrt{57}$

c) Find $\mathrm{proj}_E \mathbf{h}$. [Express your solution as linear combinations of $\mathbf{f}$ and $\mathbf{g}$.]

SOLUTION: Since $\mathbf{f}$ and $\mathbf{g}$ are orthogonal, they are an orthogonal basis for E. Thus $\mathrm{proj}_E \mathbf{h} = a\mathbf{f} + b\mathbf{g}$ for some constants a and b, that is,

$$\mathbf{h} = a\mathbf{f} + b\mathbf{g} + \mathbf{w}, \tag{2}$$

for some $\mathbf{w} \perp E$. To find a and b, as usual we take the inner product of both sides with $\mathbf{f}$ and $\mathbf{g}$ and get

$$a = \frac{\langle \mathbf{h},\, \mathbf{f}\rangle}{\|\mathbf{f}\|^2} = \frac{8}{4} = 2, \qquad b = \frac{\langle \mathbf{h},\, \mathbf{g}\rangle}{\|\mathbf{g}\|^2} = \frac{3}{1} = 3.$$

Therefore,

$$\mathrm{proj}_E \mathbf{h} = 2\mathbf{f} + 3\mathbf{g}$$

d) Find an orthonormal basis of the span of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ [Express your results as linear combinations of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$.]

SOLUTION: Since $\mathbf{f}$ and $\mathbf{g}$ are orthogonal and, from equation (2), $\mathbf{w}$ is orthogonal to both $\mathbf{f}$ and $\mathbf{g}$, we find that $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{w}$ are an orthogonal bases. To get an orthonormal basis we need only normalize these. From (2),

$$\|\mathbf{h}\|^2 = \|2\mathbf{f}\|^2 + \|3\mathbf{g}\|^2 + \|\mathbf{w}\|^2$$

so $\|\mathbf{w}\|^2 = 50 - 4\cdot 4 - 9\cdot 1 = 25$. Therefore an orthonormal basis is

$$e_1 := \tfrac{1}{2}\mathbf{f}, \qquad e_2 := \mathbf{g}, \qquad e_3 := \tfrac{1}{5}\mathbf{w} = \tfrac{1}{5}(\mathbf{h} - 2\mathbf{f} - 3\mathbf{g}).$$

7. [LIKE BRETSCHER, SEC. 5.5 #26 & 28]. Use the inner product $\langle f,\, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Define

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x \le 0, \\ 1 & \text{if } \ 0 < x \le \pi, \end{cases}$$

and extend f to all of $\mathbb{R}$ as period is with period 2π: $f(x+2\pi) = f(x)$. This is called a *square wave*.

a) Compute the first N terms in the Fourier Series

$$f(x) = A_0 + \sum_{k=1}^{N} [A_k \cos kx + B_k \sin kx] + R_N(x), \tag{3}$$

where the remainder, $R_N(x)$, is orthogonal to 1, $\cos kx$, $\sin \ell x$, $k, \ell = 1, 2, \ldots, N$.

Solution: We use that with this inner product, the functions

$$1, \qquad \cos kx, \qquad \text{and} \qquad \sin \ell x, \qquad k, \ell = 1, 2, 3, \ldots$$

are orthogonal with $\|1\|^2 = 2\pi$, and $\|\cos kx\|^2 = \|\sin \ell x\|^2 = \pi$.
Then, taking the inner product of equation (3) with the $\cos jx$'s and $\sin_\ell$'s we obtain

$$A_0 = \frac{\langle f,\, 1\rangle}{2\pi}, \qquad A_k = \frac{\langle f,\, \cos kx\rangle}{\pi}, \qquad B_k = \frac{\langle f,\, \sin kx\rangle}{\pi}.$$

Since our function $f(x)$ is odd, by Problem 3 we know that $A_k = 0$ for $k = 0, 1, 2, 3. \ldots$ and

$$\begin{aligned} B_k &= -\int_{-\pi}^{0} \frac{\sin kx}{\pi}\, dx + \int_0^{\pi} \frac{\sin kx}{\pi}\, dx \\ &= 2\int_0^{\pi} \frac{\sin kx}{\pi}\, dx = \left.\frac{-2\cos kx}{\pi}\right|_0^{\pi} \\ &= \frac{2}{\pi}[1 - (-1)^k] = \begin{cases} \frac{4}{k\pi} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases} \end{aligned}$$

Therefore, for $N = 2K + 1$,

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2K+1)x}{2K+1}\right] + R_{2K+1}(x),$$

where R_N isorthogonal to the preceeding terms.
See `http://mathworld.wolfram.com/FourierSeriesSquareWave.html` for an interesting graph of how this Fourier series converges to the square wave.

b) Apply the Pythagorean Theorem 5.5.6 (page 343) to your answer.

Solution: Because the terms in equation (3) are orthogonal, the Phythagoread theorem gives

$$\|f\|^2 = 2\pi|A_0|^2 + \pi\sum_{k=1}^{N}\left[|A_k|^2 + |B_k|^2\right] + \|R_N\|^2.$$

Applied to this example, where $\|f\|^2 = 2\pi$ and assuming (without proof here that $\|R_N\| \to 0$) it gives:

$$2\pi = \pi\frac{16}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right],$$

that is,

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8},$$

which one would not likely guess.

8. Compute the determinant of the upper triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

[Do the cases $n = 2$ and $n = 3$ first.]

SOLUTION: Expamding by minors using the first column gives

$$\det A = a_{11}(-1)^{1+1} \det \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Repeating this we find that $\det A = a_{11}a_{22}\cdots a_{nn}$, so for an upper (or lower) triangular matrix, the determinant is the product of the diagonal elements.

9. The $n \times n$ matrices A and B are *similar* if there is and invertible $n \times n$ matrix S so that $B = SAS^{-1}$. If A and B are similar, show that $\det B = \det A$.

SOLUTION: Since for any $n \times n$ matrices $\det(MN) = (\det M)(\det N) = \det(NM)$, then

$$\det B = \det(SAS^{-1}) = \det(S)\det(AS^{-1}) = \det S \det S^{-1} \det(A) = \det A.$$

[Last revised: March 9, 2014]