Problem Set 4

DUE: In class Thursday, Oct. 4 *Late papers will be accepted until 1:00 PM Friday.*

In addition to the problems below, you should also know how to solve the following problems from the text. Most are simple mental exercises. These are *not* to be handed in.

Sec. 3.1, #2, 4, 13, 15, 30
Sec. 3.2 #1, 2, 9. 10, 11, 19, 27
Sec. 3.3 #1, 2, 28, 32, 35, 37, 39
Sec. 3.4 #1, 2, 3, 10, 20, 37
Sec. 4.1 # 1-5, 8, 10, 25, 26, 48, 50

1. Which of the following sets of vectors are bases for $\mathbb{R}^2$?

   a). $\{(0, 1),\ (1, 1)\}$
   b). $\{(1, 0),\ (0, 1),\ (1, 1)\}$
   c). $\{(1, 0),\ (-1, 0\}$
   d). $\{(1, 1),\ (1, -1)\}$
   e). $\{((1, 1),\ (2, 2)\}$
   f). $\{(1, 2)\}$

   SOLUTION: a), c), and d) are bases for $\mathbb{R}^2$, b) linearly dependent, e) linearly dependent and don't span, f) doesn't span.

2. For which real numbers c do the vectors: $(c, 1, 1)$, $(1, c, 1)$, $(1, 1, c)$, *not* form a basis of $\mathbb{R}^3$? For each of the values of c that you find, what is the dimension of the subspace of $\mathbb{R}^3$ that they span?

   SOLUTION: Since $\mathbb{R}^3$ has dimension 3, we need only check when these vectors are linearly dependent, that is, can we find numbers x_1, x_2, x_3 not all zero so that

   $$x_1(c, 1, 1) + x_2(1, c, 1) + x_3(1, 1, x) = 0,$$

   that is,

   $$\begin{aligned} cx_1 + x_2 + x_3 &= 0 \\ x_1 + cx_2 + x_3 &= 0 \\ x_1 + x_2 + cx_3 &= 0 \end{aligned}$$

   Adding these equations we get $(c + 2)(x_1 + x_2 + x_3) = 0$. Thus, if $c = -2$, we immediately find these vectors are linearly dependent. Since the vectors $(-2, 1, 1)$ and $(1, -2, 1)$ are clearly linearly independent, in this case these vectors span a 2 dimensional space.

   If $c \neq 2$, then adding the equations we find that $x_1 + x_2 + x_3 = 0$. Comparing this with the equations we find that if $c \neq 1$, then $x_1 = x_2 = x_3 = 0$. However if $c = 1$ then the three vectors only span the one dimensional space of vectors of the form $a(1,\ 1,\ 1)$.

3. Compute the dimension and find bases for the following linear spaces.

a) [SEE BRETSCHER, SEC. 4.1 #25]. Quartic polynomials $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ with the property that $p(2) = 0$ and $p(3) = 0$.

SOLUTION: Since $p(x)$ is zero at $x = 2$ and $x = 3$, we can factor $x - 2$ and $x - 3$ from $p(x)$ so $p(x)$ has the form

$$p(x) = (x - 2)(x - 3)(a + bx + cx^2).$$

Thus the polynomials

$$p_1(x) = (x - 2)(x - 3), \qquad p_2(x) = (x - 2)(x - 3)x, \qquad p_3(x) = (x - 2)(x - 3)x^2$$

form a basis. The dimension of this space is therefore 3.

b) [SEE BRETSCHER, SEC. 4.1 #8] Real upper triangular 3×3 matrices (first, show that this is a linear space).

SOLUTION: As a warm-up, first the case of 2×2 upper triangular matrices. These have the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The three matrices on the right form a basis and the dimension of this space is 3.

Now the 3×3 case: Each of these matrices have the form $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$. It should be clear that the following is a basis

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

The dimension of this space is 6.

c) The space of linear maps $L : \mathbb{R}^5 \to \mathbb{R}^3$ whose kernels contain $(0, 2, -3, 0, 1)$.

SOLUTION: As in Homework Set 3 #8, Let V_1, $V_2, \ldots$, V_5 be the columns of L. Then $2V_2 - 3V_3 + V_5 = 0$. We solve this, say, for $V_5 = -2V_2 + 3V_3$. This tells us that L has the form

$$L = \left(\begin{pmatrix} V_1 \end{pmatrix} \begin{pmatrix} V_2 \end{pmatrix} \begin{pmatrix} V_3 \end{pmatrix} \begin{pmatrix} V_4 \end{pmatrix} \begin{pmatrix} -2V_2 + 3V_3 \end{pmatrix} \right)$$

Each of the 4 vectors V_j, $j = 1, 2, 3, 4$, which are arbitrary, have 3 components. Therefore the space of all such maps L has dimension $4 \times 3 = 12$.

4. [SEE BRETSCHER, SEC. 3.2 #6] Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and let $W = U \cap V$. Find all possible values for the dimension of W.

   SOLUTION: Let $e_1 = (1,0,0,0,0)$, $e_2 = (0,1,0,0,0)$,..., $e_5 = (0,0,0,0,1)$ be the standard basis for $\mathbb{R}^5$ and say U is spanned by e_1 and e_2.

   If V is also spanned by e_1 and e_2 the dimension of W is 2, clearly the largest possible.

   If V is spanned by e_1 and e_3 the dimension of W is 1.

   If V is spanned by e_3 and e_4 the dimension of W is 0. They intersect only at the origin.

5. [SEE BRETSCHER, SEC. 3.2 #50] Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and define the set $W := U + V$ as the set of all vectors $w = u + v$ where $u \in U$ and $v \in V$ can be any vectors.

   a) Show that W is a linear space.

      SOLUTION: Since the sum of two vectors in U is in U and the sum of two vectors in V is also in V, then the sum of two vectors in W is also in W
      Similarly, if $\vec{w} = \vec{u} + \vec{v} \in W$, then so is $c\vec{w} = c\vec{u} + c\vec{v}$ for any scalar c.

   b) Find all possible values for the dimension of W.

      SOLUTION: We use the notation of the previous problem.
      If V is also spanned by e_1 and e_2 the dimension of W is 2, clearly the smallest possible.
      If V is spanned by e_1 and e_3 the dimension of W is 3.
      If V is spanned by e_3 and e_4 the dimension of W is 4. This is the largest possible.

6. [SEE BRETSCHER, SEC. 3.2 #42] Let $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ be orthogonal unit vectors in $\mathbb{R}^n$. Show that they must be linearly independent. [This problem s very short.]

   SOLUTION: Say

   $$a\vec{v_1} + b\vec{v_2} + c\vec{v_3} = 0$$

   for some scalars a, b, and c. We want to show that $a = b = c = 0$ Take the inner product of both sides with $\vec{v_1}$ gives $a = 0$. Similarly, $b = c = 0$.

7. Say you have k linear algebraic equations in n variables; in matrix form we write $A\vec{x} = \vec{y}$. Give a proof or counterexample for each of the following.

   a) If $n = k$ there is always *at most one* solution.

      SOLUTION: False. $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ are both counterexamples. It is true only if A is invertible.

   b) If $n > k$, given *any* $\vec{y}$. you can *always* solve $A\vec{x} = \vec{y}$.

      SOLUTION: False. Counterexamples: $A = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{smallmatrix}\right)$.

c) If $n > k$ the nullspace of A has dimension greater than zero.

SOLUTION: True. For $A\vec{x} = \vec{y}$, if there are more unknowns than equations, then the homogeneous equation $A\vec{x} = 0$ always has a a solution other than the trivial solution $\vec{x} = 0$.

d) If $n < k$ then for *some* $\vec{y}$ there is *no* solution of $A\vec{x} = \vec{y}$.

SOLUTION: True. If $A : \mathbb{R}^n \to \mathbb{R}^k$, then the dimension of the image of A is at most n. Thus, if $n < k$ then A cannot be onto.

e) If $n < k$ the *only* solution of $A\vec{x} = 0$ is $\vec{x} = 0$.

SOLUTION: False. Counterexamples: $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix}$.

8. Find a 3×3 matrix that acts on $\mathbb{R}^3$ as follows: it keeps the x_1 axis fixed but rotates the x_2 x_3 plane by 60 degrees.

SOLUTION: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}$

This problem is a bit ambiguous since the problem does not specify how you should measure the angle (beginning from the x_2 axis or x_3 axis) and the sense of the rotation (something like clockwise or counter-clockwise).

9. Give a proof or counterexample the following. In each case your answers should be brief.

a) Suppose that $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in a vector space V and $T : V \to W$ is a linear map. If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are linearly dependent, is it true that $T(\vec{u})$, $T(\vec{v})$ and $T(\vec{w})$ are linearly dependent? Why?

SOLUTION: True. Say $a\vec{u} + b\vec{v} + c\vec{w} = 0$. Then $aT(\vec{u}) + bT(\vec{v}) + cT(\vec{w}) = T(a\vec{u} + b\vec{v} + c\vec{w}) = T(0) = 0$.

b) If $T : \mathbb{R}^6 \to \mathbb{R}^4$ is a linear map, is it possible that the nullspace of T is one dimensional?

SOLUTION: Impossible. By the *Dimension Theorem* ("Rank-Nullity Theorem," p. 129, p. 165)

$$\dim \mathbb{R}^6 = \dim \ker(T) + \dim \operatorname{im}(T) \qquad \text{so} \qquad \dim \ker(T) = 6 - \dim \operatorname{im}(T).$$

Since $\operatorname{im}(T) \subset \mathbb{R}^4$, then $\dim \operatorname{im}(T) \leq 4$. Thus $\dim \ker(T) \geq 2$.

10. Find a polynomial $p(x)$ of degree at most 3 that passes through the following 4 data points in the plane $\mathbb{R}^2$: $(1, 1)$, $(2, 0)$, $(3, -1)$, and $(4, 3)$.

Solution: As in class, we follow Lagrange and use a clever basis for the space of cubic polynomials that is adapted to this problem. Call the data points (x_j, y_j), $j = 1, 2, 3, 4$. We construct cubic polynomials $p_i(x)$, $i = 1, 2, 3, 4$ so that $p_i(x_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$. Then the desired polynomial is

$$p(x) := y_1 p_1(x) + y_2 p_2(x) + y_3 p_3(x) + y_4 p_4(x).$$

For our data points,

$$p_1(x) := \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} \qquad p_2(x) := \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}$$

$$p_3(x) := \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} \qquad p_4(x) := \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$$

so

$$\begin{aligned} p(x) :=& 1\, p_1(x) + 0\, p_2(x) - 1 p_3\,(x) + 3\, p_4(x) \\ =& \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} - \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + 3\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}. \end{aligned}$$

11. [Bretscher, Sec. 3.1 #37] For the matrix $M := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ describe the kernels and images of M, M^2, and M^3 geometrically.

Solution: Let e_1, e_2, and e_3 be the standard basis for $\mathbb{R}^3$. Then $Me_1 \to 0$, $Me_2 \to e_1$, and $Me_3 \to e_2$. So the e_1 axis is the kernel of M and the plane spanned by e_1 and e_2 is the image of M.

Repeating this, $M^2 e_1 = M(M(e_1) = 0$, $M^2 e_2 = M(Me_2) = 0$, and $M^2 e_3 = M(Me_3) = e_1$ so the plane spanned by e_1 and e_2 is the kernel of M^2 while the e_3 axis is the image of M_2. $M^3 = 0$ so the kernel is all of $\mathbb{R}^3$ while the image is just the origin (which has dimension 0.

12. [Bretscher, Sec. 3.2 #46] Find a basis for the kernel of the matrix

$$M := \begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \end{pmatrix}.$$

Justify your answer carefully; that is, explain how the vectors you found are both linearly independent and span the kernel.

Solution: To find the kernel we solve the homogeneous equation $M\vec{x} = 0$, that is

$$\begin{aligned} x_1 + 2x_2 \qquad + 3x_4 + \quad 5x_5 =& 0 \\ x_3 + 4x_4 + \quad 6x_5 =& 0 \end{aligned}$$

We solve the first equation for, say $x_1 = -2x_2 - 3x_4 - 5x_5$ and the second for $x_3 = -4x_4 - 6x_5$. Thus, for any choice of x_2, x_4 and x_5

$$\vec{x} = \begin{pmatrix} -2x_2 - 3x_4 - 5x_5 \\ x_2 \\ -4x_4 - 6x_5 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{pmatrix} x_5.$$

The three column vectors on the right are a basis for the kernel of M.

13. Compute the rank (dimension of the image) of each of the following matrices.

$$a).\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad b).\ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad c).\ \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 7 \\ 0 & 0 & 2 \end{pmatrix} \qquad d).\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{pmatrix} \qquad e).\ \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

SOLUTION: The image of a matrix is the span of the columns, that is, all possible linear combinations of the columns.

In a) and b) the image is all multiples of the second column, so the image is one dimensional.

In c). The matrix is upper triangular with non-zero diagonal elements. Thus the three columns are linearly independent so the dimension of the image is 3.

Alternate method: If we call the matrix A, another way to see this is to find the vectors $\vec{y} = (y_1, y_2, y_3)$ for which one can solve $A\vec{x} = \vec{y}$, that is,

$$\begin{aligned} x_1 + 2x_2 \qquad\qquad &= y_1 \\ -x_2 \quad +7x_3 &= y_2 \\ 2x_3 &= y_3 \end{aligned}$$

This is an upper triangular system. Solve the last equation for x_3, then the second equation for x_2 and finally the first equation for x_1. Thus, given any $\vec{y} \in \mathbb{R}^2$ there is a (unique) solution so the dimension of the image is 3.

In d). The matrix is upper triangular. The first column is zero but the last two columns are clearly linearly independent. The dimension of the image is 2.

Alternate method: Solve $A\vec{x} = \vec{y}$, as in c).

In e). Since the third component of each of the columns is zero. The first two columns are linearly independent (in fact any two of them are linearly independent) do the dimension of the image is 2.

Alternate method: Solve $A\vec{x} = \vec{y}$, as in c).

14. Compute the rank (dimension of the image) of each of the following matrices:

$$A := \begin{pmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -5 \end{pmatrix}, \qquad B := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solutions: Since the image is the span of the columns of these matrices, we use the procedure of the previous problem. Since all of these matrices are upper triangular, the approach of directly solving the equations $M\vec{x} = \vec{y}$ (where M is A, B, or C) is particularly transparent and gives

$$\operatorname{rank}(A) = 3, \qquad \operatorname{rank}(B) = 5, \qquad \operatorname{rank}(C) = 3.$$

15. [Bretscher, Sec. 3.3 #30] Find a basis for the subspace of $\mathbb{R}^4$ defined by the equation $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

Solution: Solve this for, say, $x_2 = 2x_1 + 2x_3 + 4x_4$. Then a vector $\vec{x}$ is in the subspace if (and only if) for any choice of x_1, x_3, and x_4

$$\vec{x} = \begin{pmatrix} x_1 \\ 2x_1 + 2x_3 + 4x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} 0 \\ 4 \\ 0 \\ 1 \end{pmatrix} x_4.$$

The three column vectors on the right are a basis for this subspace: dimension is 3.

16. [Bretscher, Sec. 4.1 #51] Find all solutions $f(x)$ of the differential equation

$$f'' - 7f' + 12f = 0.$$

Solution: Let $Lf := f'' - 7f' + 12f$. Observe that $Le^{rx} = (r^2 - 7r + 12)e^{rx}$. Consequently, if r is a root of the quadratic polynomial $r^2 - 7r + 12$, then e^{rx} is a solution of the homogeneous equation $Lf = 0$. The roots are $r = 3$ and $r = 4$, so both e^{3x} and e^{4x} are solutions of $Lf = 0$. By linearity, any linear combination of these is also a solution:

$$f(x) = ae^{3x} + be^{4x}.$$

This is the general solution (a fact we have not yet proved) so the dimension of the kernel of L is two. The proof of the missing fact is an easy consequence of the following more general

Theorem *Let $A(x)$ be a square matrix whose elements depend continuously on x. If the vector $\vec{u}(x)$ is a solution of the homogeneous first order linear ordinary differential equation (ode) $\vec{u}' + A(x)\vec{u} = 0$ with $\vec{u}(0) = 0$, then $\vec{u}(x) \equiv 0$.*

We'll prove this in class soon. It is not difficult.

REMARK: Bretscher, Sec. 4.1 #58 was done in class. (This is not a homework problem.)

Bonus Problem

[Please give this directly to Professor Kazdan]

1-B [BRETSCHER, SEC. 3.3 #64] Two subspaces V and W of $\mathbb{R}^n$ are called *complements* if any vector $\vec{x} \in \mathbb{R}^n$ can be expressed uniquely as $\vec{x} = \vec{v} + \vec{w}$, where $\vec{v} \in V$ and $\vec{w} \in W$. Show that V and W are complements if (and only if) $V \cap W = 0$ and $\dim V + \dim W = n$.

[Last revised: December 22, 2012]