Math 312, Fall 2012 Jerry L. Kazdan

Problem Set 2

DUE: In class Thursday, Sept. 20. *Late papers will be accepted until 1:00 PM Friday.*

Lots of problems. Most are really short.

In addition to the problems below, you should also know how to solve the following problems from the text. Most are simple mental exercises. These are *not* to be handed in.

Sec. 1.1 #32, # 35
Sec. 1.2 #2, #12
Sec. 1.3 #29, #34, #36, #37, #47, #48, #52, #53, #56, #59, #65
p. 39 #39
Sec. 2.1 #1, #2, #3, #4, #6, #19, #20, #22, #25, #32, #39, #46
Sec. 2.2 #2, #4, #8
Sec. 2.3 #4, #10, #11, #12, #13, #14, #55, #58

1. [Bretscher, Sec. 1.2 #32] Find the polynomial $f(t)$ of degree 3 such that $f(1) = 1$, $f(2) = 5$, $f'(1) = 2$, $f'(2) = 9$. Graph this polynomial.

2. [Bretscher, Sec.2.1 #13]
   a) Let $A := \left(\begin{smallmatrix} 1 & 2 \\ c & 6 \end{smallmatrix}\right)$. With your bare hands (not using anything about determinants) show that A is invertible if and only if $c \neq 3$.
   b) Let $M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. With your bare hands (not using anything about determinants) show that M is invertible if and only if $ad - bc \neq 0$. [*Hint:* Treat the cases $a \neq 0$ and $a = 0$ separately.]

3. [Bretscher, Sec.2.2 #6] (Review from Math 240) Let $\mathcal{L}$ be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $(2, 1, 2)$. Find the orthogonal projection of the vector $(1, 1, 1)$ onto $\mathcal{L}$.

4. [Bretscher, Sec.2.2 #10] Let $\mathcal{L}$ be the line in $\mathbb{R}^2$ that consists of all scalar multiples of the vector $(4, 3)$. Find the matrix of the orthogonal projection onto this line $\mathcal{L}$.

5. [Bretscher, Sec.2.2 #17] Let $A := \left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$, where $a^2 + b^2 = 1$. Find two perpendicular non-zero vectors $\vec{v}$ and $\vec{w}$ so that $A\vec{v} = \vec{v}$ and $A\vec{w} = -\vec{w}$ (write the entries of $\vec{v}$ and $\vec{w}$ in terms of a and b). Conclude that thinking of A as a linear map it is an orthogonal reflection across the line $\mathcal{L}$ spanned by $\vec{v}$.

6. [Bretscher, Sec.2.2 #31] Find a nonzero 3×3 matrix A so that $A\vec{x}$ is perpendicular to $\vec{v} := (1, 2, 3)$ for all vectors $\vec{x} \in \mathbb{R}^3$.

7. [Bretscher, Sec.2.3 #19] Find all the matrices that commute with $A := \left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$.

8. [Bretscher, Sec.2.3 #48]

   a) Geometrically interpret the matrix $M := \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$. What is M^{-1}, both geometrically and computationally?

   b) If $A := \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, compute AB and A^{10}.

   c) Find a 2×2 matrix A so that $A^{10} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

9. Which of the following sets are linear spaces?

   a) $\{X = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0\}$

   b) The set of solutions x of $Ax = 0$, where A is an $m \times n$ matrix.

   c) The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)\, dx = 0$.

   d) The set of solutions $y = y(t)$ of $y'' + 4y' + y = 0$.

10. Proof or counterexample. In these L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, so its representation will be as a 2×2 matrix.

   a) If L is invertible, then L^{-1} is also invertible.

   b) If $L\vec{v} = 5\vec{v}$ for all vectors $\vec{v}$, then $L^{-1}\vec{w} = (1/5)\vec{w}$ for all vectors $\vec{w}$.

   c) If L is a rotation of the plane by 45 degrees *counterclockwise*, then L^{-1} is a rotation by 45 degrees *clockwise*.

   d) If L is a rotation of the plane by 45 degrees counterclockwise, then L^{-1} is a rotation by 315 degrees counterclockwise.

   e) The zero map ($0\vec{v} := 0$ for all vectors $\vec{v}$) is invertible.

   f) The identity map ($I\vec{v} := \vec{v}$ for all vectors $\vec{v}$) is invertible.

   g) If L is invertible, then $L^{-1}0 = 0$.

   h) If $L\vec{v} = 0$ for some non-zero vector $\vec{v}$, then L is not invertible.

   i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: $L = L^{-1}$.

11. Think of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as mapping one plane to another.

   a) If two lines in the first plane are parallel, show that after being mapped by A they are also parallel – although they might coincide.

   b) Let Q be the unit square: $0 < x < 1, 0 < y < 1$ and let Q' be its image under this map A. Give a geometric argument to show that the area$(Q') = |ad - bc|$. [More generally, the area of any region is magnified by $|ad - bc|$, which is called the *determinant* of A.]

12. Let A be a matrix, not necessarily square. Say $\vec{v}$ and $\vec{w}$ are particular solutions of the equations $A\vec{v} = \vec{y}_1$ and $A\vec{w} = \vec{y}_2$, respectively, while $\vec{z} \neq 0$ is a solution of the homogeneous equation $A\vec{z} = 0$. Answer the following in terms of $\vec{v}$, $\vec{w}$, and $\vec{z}$.

   a) Find some solution of $A\vec{x} = 3\vec{y}_1$.

   b) Find some solution of $A\vec{x} = -5\vec{y}_2$.

   c) Find some solution of $A\vec{x} = 3\vec{y}_1 - 5\vec{y}_2$.

   d) Find another solution (other than $\vec{z}$ and 0) of the homogeneous equation $A\vec{x} = 0$.

   e) Find *two* solutions of $A\vec{x} = \vec{y}_1$.

   f) Find another solution of $A\vec{x} = 3\vec{y}_1 - 5\vec{y}_2$.

   g) If A is a square matrix, for any given vector $\vec{w}$ can one always find at least one solution of $A\vec{x} = \vec{w}$? Why?

13. Let V be the linear space of smooth real-valued functions and $L : V \to V$ the linear map defined by $Lu := u'' + u$.

   a) Compute $L(e^{2x})$ and $L(x)$.

   b) Find particular solutions of the inhomogeneous equations

$$a).\ u'' + u = 7e^{2x}, \qquad b).\ w'' + w = 4x, \qquad c).\ z'' + z = 7e^{2x} - 3x$$

14. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$, so $BA : \mathbb{R}^3 \to \mathbb{R}^3$ and $AB : \mathbb{R}^2 \to \mathbb{R}^2$.

   a) Why must there be a non-zero vector $\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = 0$.

   b) Show that the 3×3 matrix BA can *not* be invertible.

   c) Give an example showing that the 2×2 matrix AB might be invertible.

Bonus Problem

[Please give this directly to Professor Kazdan]

1-B Let $\mathcal{P}_N$ be the linear space of polynomials of degree at most N and $L : \mathcal{P}_N \to \mathcal{P}_N$ the linear map defined by $Lu := au'' + bu' + cu$, where a, b, and c are constants. Assume $c \neq 0$.

   a) Compute $L(x^k)$.

   b) Show that nullspace (=kernel) of $L : \mathcal{P}_N \to \mathcal{P}_N$ is 0.

   c) Show that for every polynomial $q(x) \in \mathcal{P}_N$ there is one (and only one) solution $p(x) \in \mathcal{P}_N$ of the ODE $Lp = q$.

   d) Find some solution $v(x)$ of $v'' + v = x^2 - 1$.

1[Last revised: September 16, 2012]