Problem Set 10

DUE: Never.

1. [BRETSCHER, SEC. 7.1 #12] Find the eigenvalues and eigenvectors of $A := \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$.

   SOLUTION: Since A is lower triangular, the eigenvalues are clearly $\lambda_1 = 2$ and $\lambda_2 = 4$. The corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2. [BRETSCHER, P. 318,SEC. 7.2 #28] The isolated Swiss town inhabited by 1,200 families had only one grocery store owned by Mr. and Ms. Wipf. Each family made a weekly shopping trip. Recently a fancier (and cheaper) chain store, Migros opened. It is anticipated that 20% of the Wipf shoppers each week will switch to Migros the following week. However some people who switch will miss the personal service and switch back to Wipf the following week. Let w_k be the number of families who shop at Wipf's and m_k the number who shop at Migros k weeks after Migros opened, so $w_0 = 1,200$ and $m_0 = 0$. This describes a Markov Chain whose state is described by the vector

$$\vec{x}_k := \begin{pmatrix} w_k \\ m_k \end{pmatrix}.$$

   a) Find a 2×2 transition matrix A so that $\vec{x}_{k+1} = A\vec{x}_k$.

      SOLUTION: $A = \begin{pmatrix} .8 & .1 \\ .2 & .9 \end{pmatrix}$

   b) How many families will shop at each store after k weeks? Give closed formulas.

      SOLUTION: Since $\vec{x}_k = A^k\vec{x}_0$, we need to compute A^k. For this we diagonalize A. Since A is the transition matrix of a Markov process, one of the eigenvalues is $\lambda_1 = 1$ For the eigenvalues, we see that trace$(A) = 1.7$, then $\lambda - 2 = .7$. The corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. If we use the change of coordinates $S := \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$, then A is similar to the diagonal matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & .7 \end{pmatrix}$, $S^{-1}AS = D$ so $A^k = SD^kS^{-1}$. Since $S^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$, by a straightforward computation

$$\vec{x}_k = SD^kS^{-1}\vec{x}_0 = 400\begin{pmatrix} 1 + 2(.7)^k \\ 2 - 2(.7)^k \end{pmatrix} \to \begin{pmatrix} 400 \\ 800 \end{pmatrix}.$$

   c) The Wipfs expect that they must close down when they have fewer than 250 customers a week. When does that happen?

      SOLUTION: From the above, $w_k = 400(1 + 2(.7)^k) \searrow 400$, so $w_k > 400$ which is larger than the critical 350.

3. If $\vec{v}$ is an eigenvector of the matrix A, show that it is also an eigenvector of $A + 37I$. What is the corresponding eigenvalue?

SOLUTION: $(A+37I)\vec{v} = A\vec{v} + 37\vec{v} = (\lambda + 37)\vec{v}$ so the corresponding eigenvalue is $\lambda + 37$.

4. Let A be an invertible matrix. Show that $\lambda = 0$ cannot be an eigenvalue.

   Conversely, if a (square) matrix is not invertible, show that $\lambda = 0$ is an eigenvalue.

   SOLUTION: If $\lambda = 0$ were an eigenvalue, then there is a $\vec{v} \neq 0$ so that $A\vec{v} = 0$. But then A would not be one-to-one.

   Conversely, if 0 is not an eigenvalue, then the kernel of A is just the zero vector.

5. Let $z = x + iy$ be a complex number. For which real numbers x, y is $|e^z| < 1$?

   SOLUTION: Since $e^z = e^{x+iy} = e^x e^{iy}$ and $|e^{iy} = 1|$, then $|e^z| = e^x$. This is less than 1 for all $x < 0$.

6. Let M be a 4×4 matrix of real numbers. If you know that both $1 + 2i$ and $2 - i$ are eigenvalues of M, is M diagonalizable? Proof or counterexample.

   SOLUTION: Since the matrix has only real elements, the complex conjugates. $1 - 2i$ and $2 + i$ of these complex numbers are also eigenvalues. Thus M has 4 distinct eigenvalues and thus is diagonalizable.

7. Let A and B be $n \times n$ real positive definite matrices and let $C := tA + (1-t)B$. If $0 \le t \le 1$, show that C is also positive definite. [This is simple. No "theorems" are needed.]

   SOLUTION: Since $0 \le t \le 1$, if $\vec{x} \neq 0$ then

$$\langle \vec{x}, C\vec{x} \rangle = \langle \vec{x}, [tA + (1-t)B]\vec{x} \rangle = \quad t\langle \vec{x}, A\vec{x} \rangle + (1-t)\langle \vec{x}, B\vec{x} \rangle > 0.$$

8. Let $A := \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

   a) Find the eigenvalues and eigenvectors of A.

SOLUTION:

$$\begin{aligned}\det(A-\lambda I) =&(1-\lambda)\det\begin{pmatrix}3-\lambda & 0 & 1\\ 0 & 1-\lambda & 0\\ 1 & 0 & 3-\lambda\end{pmatrix}\\ =&(1-\lambda)^2\det\begin{pmatrix}3-\lambda & 1\\ 1 & 3-\lambda\end{pmatrix}\\ =&(1-\lambda)^2\left[(3-\lambda)^2-1\right]=(1-\lambda)^2(\lambda-2)(\lambda-4).\end{aligned}$$

Thus the eigenvalues are $\lambda_1=1$, $\lambda_2=1$, $\lambda_3=2$, and $\lambda_4=4$.

To find the eigenvectors corresponding to $\lambda=1$ we want non-trivial solutions $\vec{x}:=(x_1,\,x_2,\,x_3,\,x_4)$ of the equations $(A-I)\vec{x}=0$, that is,

$$\begin{aligned}2x_1+0x_2+\ \ x_3+0x_4=&0\\ x_1+0x_2+2x_3+0x_4=&0.\end{aligned}$$

These imply $x_1=x_3=0$ but x_2 and x_4 can be anything; every point of the form $(0,\,x_2,\,0,\,x_4)$ is an eigenvector with eigenvalue 1. We pick a simple orthonormal basis of this space as the eigenvectors: $\vec{v}_1=(0,1,0,0)$ and $\vec{v}_2=(0,0,0,1)$.

For the eigenvalue $\lambda_3=2$ it is straightforward to find the eigenvector $\vec{v}_3=(1,0,-1,0)$ while $\lambda_4=4$ we get $\vec{v}_4=(1,0,1,0)$.

b) Find an orthogonal transformation R so that $R^{-1}AR$ is a diagonal matrix.

SOLUTION: The eigenvectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ are already orthogonal, mainly because for a symmetric matrix such as A, they are eigenvectors corresponding to distinct eigenvalues. However, the orthogonality of $\vec{v}_1$ and $\vec{v}_2$ was because we could have chosen any linearly independent vectors in this eigenspace so for simplicity we chose orthogonal vectors. To make these orthonormal we need only adjust them so that they are unit vectors. Only $\vec{v}_3$ and $\vec{v}_4$ need to be fixed. We replace them by $\vec{w}_3:=\vec{v}_3/\sqrt{2}$ and $\vec{w}_4:=\vec{v}_4/\sqrt{2}$ and then use $\vec{v}_1$, $\vec{v}_2$, $\vec{w}_3$, and $\vec{w}_4$ as the columns of the matrix R.

9. If $A=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$, solve $\dfrac{d\vec{x}}{dt}=A\vec{x}$ with initial condition $\vec{x}(0)=\begin{pmatrix}1\\0\end{pmatrix}$.

SOLUTION: We first diagonalize A. Its eigenvalues and corresponding eigenvectors are $\lambda_1=3$, $\lambda_2=-1$, $\vec{v}_1=\begin{pmatrix}1\\1\end{pmatrix}$, and $\vec{v}_2=\begin{pmatrix}1\\-1\end{pmatrix}$. Use the eigenvectors of A as the columns of the change of coordinates matrix $S=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$. Then $A=SDS^{-1}$, where $D=\begin{pmatrix}3 & 0\\ 0 & -1\end{pmatrix}$.

Thus

$$\frac{d\vec{x}}{dt}=SDS^{-1}\vec{x}\qquad\text{so}\qquad\frac{dS^{-1}\vec{x}}{dt}=DS^{-1}\vec{x}.$$

Make the change of variable $\vec{y} := S^{-1}\vec{x}$. Then $\dfrac{d\vec{y}}{dt} = D\vec{y}$, that is,

$$\begin{aligned} y_1' &= 3y_1 \\ y_2' &= -y_2. \end{aligned}$$

Because D is a diagonal matrix, this system of differential equations is *uncoupled*; the first involves only y_1 and the second only y_2. The solution is clearly $y_1(t) = \alpha e^{3t}$, $y_2(t) = \beta e^{-t}$ for any constants α and β. Thus we now know that

$$\vec{x}(t) = S\vec{y}(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha e^{3t} \\ \beta e^{-t} \end{pmatrix} = \begin{pmatrix} \alpha e^{3t} + \beta e^{-t} \\ \alpha e^{3t} - \beta e^{-t} \end{pmatrix}.$$

As the final step we pick α and β so that $\vec{x}(t)$ satisfies the initial condition on $\vec{x}(0)$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{x}(0) = \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}$$

so $\alpha = \beta = 1/2$. In summary:

$$\vec{x}(t) = \tfrac{1}{2} \begin{pmatrix} e^{3t} + e^{-t} \\ e^{3t} - e^{-t} \end{pmatrix}$$

10. Let A and B be any 3×3 matrices. Show that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$. [This is also true for $n \times n$ matrices.]

SOLUTION: If $A = \bigl(a_{ij}\bigr)$ and $B = \bigl(b_{ij}\bigr)$, let $C = AB$. Say $(AB)_{ik}$ is the ik element in the matrix AB. Then the rule for matrix multiplication gives

$$(AB)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

Consequently

$$\operatorname{trace}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \Bigl(\sum_{j=1}^{n} a_{ij} b_{ji} \Bigr).$$

Similarly

$$\operatorname{trace}(BA) = \sum_{i=1}^{n} (BA)_{ii} = \sum_{i=1}^{n} \Bigl(\sum_{j=1}^{n} b_{ij} a_{ji} \Bigr).$$

Since i and j are dummy indices of summation, interchanging them in the formula for the trace of BA shows that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$.

Use this to give another proof that if the matrices M and Q are similar, then $\operatorname{trace}(M) = \operatorname{trace}(Q)$.

SOLUTION: If M and Q are similar, then there is an invertible matrix S so that $Q = S^{-1}MS$. Therefore

$$\operatorname{trace} Q = \operatorname{trace}(S^{-1}MS) = \operatorname{trace}\bigl(S^{-1}(MS)\bigr) = \operatorname{trace}\bigl((MS)S^{-1}\bigr) = \operatorname{trace} M.$$

11. Let $A := \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix}$.

a) Compute A^{50}.

SOLUTION: Since A is the transition matrix for a Markov chain, $\lambda_1 = 1$. Since the trace of A is $3/4$, then $\lambda_2 = -1/4$. By a routine computation, the corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Let $S = \left(\begin{smallmatrix} 2 & 1 \\ 3 & -1 \end{smallmatrix}\right)$ be the matrix whose columns are these eigenvectors and let $D = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1/4 \end{smallmatrix}\right)$. Then $A = SDS^{-1}$. Therefore

$$\begin{aligned} A^{50} = SD^{50}S^{-1} &= \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-1/4)^{50} \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{pmatrix} \\ &= \tfrac{1}{5} \begin{pmatrix} 2 + 3/4^{50} & 2 - 2/4^{50} \\ 3 - 3/4^{50} & 3 + 2/4^{50} \end{pmatrix}. \end{aligned}$$

b) Let $P_0 := \begin{pmatrix} p \\ q \end{pmatrix}$ where $p > 0$ and $q > 0$ with $p + q = 1$. Compute $A^{50}P_0$. What do you suspect $\lim_{k\to\infty} A^k P_0 =?$.

SOLUTION: $A^{50}P_0 = \frac{1}{5} \begin{pmatrix} 2 + (5p-2)/4^{50} \\ 3 - (5p-2)/4^{50} \end{pmatrix}$.

One sees that $A^k P_0 \to \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}$.

c) Note that A is the transition matrix of a Markov process. What do you suspect is the long-term stable state? Verify your suspicion.

SOLUTION: The above limiting is exactly the probability vector associated with the long-term stable state.

12. Let A be a 3×3 matrix whose eigenvalues are $-1 \pm i$ and -2. If $\vec{x}(t)$ is a solution of $\dfrac{d\vec{x}}{dt} = A\vec{x}$, show that $\lim_{t\to\infty} \vec{x}(t) = 0$ independent of the initial value $\vec{x}(0)$.

SOLUTION: Since the eigenvalues of A are distinct, A can be diagonalized. Thus for some invertible matrix S we know that $A = SDS^{-1}$, where D is a diagonal matrix consisting of the eigenvalues λ_j of A. Thus $\dfrac{d\vec{x}}{dt} = SDS^{-1}\vec{x}$. Just as in Problem 9 above, multiply both sides on the left by S^{-1} and let $\vec{y}(t) = S^{-1}\vec{x}(t)$. Then $\dfrac{d\vec{y}}{dt} = D\vec{y}$. Therefore the components $y_j(t)$ of $\vec{y}$ satisfy the uncoupled equations $y_j'(t) = \lambda_j y_j$ whose solutions are $y_j(t) = c_j e^{\lambda_j t}$ Since the real parts of the eigenvalues are all negative, by Problem 5 above, then $|y_j(t)| \to 0$ as $t \to \infty$. However, $\vec{x}(t) = S\vec{y}(t)$ so also $\|\vec{x}(t)\| \to 0$ as $t \to \infty$.

13. a) If $B := \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$, find a self adjoint matrix Q so that $Q^2 = B$. [This should be obvious.]

SOLUTION: Let $Q = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

b) If $A := \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$, find a self adjoint matrix P so that $P^2 = A$.

SOLUTION: The eigenvalues of A are $\lambda_1 = 9$ and $\lambda_2 = 1$. Therefore A is similar, by an orthogonal transformation R, to the matrix B in part a):

$$A = RBR^{-1} = RQ^2R^{-1} = \left(RQR^{-1}\right)\left(RQR^{-1}\right) = P^2,$$

where $P = RQR^{-1}$. Note that P is self-adjoint (since $R^{-1} = R^*$) and positive definite since we take the positive square root of the eigenvalues of A.
In this specific problem, to determine R explicitly we need the eigenvectors of A. For $\lambda_1 = 9$ the eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ while for $\lambda_2 = 1$ the eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
To get the columns of the orthogonal transformation R we replace the $\vec{v}_j$ by unit vectors, so

$$R := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Thus, $A = P^2$ where

$$P := RQR^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The identical proof shows that every positive definite real matrix has a unique positive definite "square root."

14. Let $A := \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$. Solve $\dfrac{d^2\vec{x}(t)}{dt^2} + A\vec{x}(t) = 0$ with $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{x}\,'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
[REMARK: If A were the diagonal matrix $\begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$, then this problem would have been simple.]

SOLUTION: The procedure is essentially identical to that used in Problem 9. Begin by diagonalizing A. We have already done the hard work in the previous problem: $A = RBR^{-1}$, where $B = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$. The differential equation is then

$$\frac{d^2\vec{x}(t)}{dt^2} + RBR^{-1}\vec{x}(t) = 0.$$

Multiply this on the left by R^{-1} and make the change of variable $\vec{y}(t) := R^{-1}\vec{x}(t)$. The differential equation then is

$$\frac{d^2\vec{y}(t)}{dt^2} + B\vec{y}(t) = 0.$$

Since B is a diagonal matrix, this system of equations is *uncoupled*:

$$y_1'' + 9y_1 = 0 \qquad \text{and} \qquad y_2'' + y_2 = 0.$$

The solution to these are

$$y_1(t) = a\cos 3t + b\sin 3t, \qquad \text{and} \qquad y_2(t) = c\cos t + d\sin t,$$

where a, b, c, and d are any constants. Now that we know $\vec{y}(t)$ we can get $\vec{x}(t)$ from

$$\begin{aligned}\vec{x}(t) = R\vec{y} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a\cos 3t + b\sin 3t \\ c\cos t + d\sin t \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} a\cos 3t + b\sin 3t + c\cos t + d\sin t \\ a\cos 3t + b\sin 3t - c\cos t - d\sin t \end{pmatrix}\end{aligned}$$

At this point, we absorb the $\sqrt{2}$ factor into the still unknown coefficients a, b, c, and d which we will determine using the initial conditions.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{x}(0) = \begin{pmatrix} a+c \\ a-c \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{x}\,'(0) = \begin{pmatrix} 3b+d \\ 3b-d \end{pmatrix}.$$

Thus $a = c = 1/2$ and $b = d = 0$ so

$$\vec{x}(t) = \frac{1}{2}\begin{pmatrix} \cos 3t + \cos t \\ cos 3t - \cos t \end{pmatrix}.$$

Note that the same technique works if A is not self-adjoint – as long as it is diagonalizable. Of course in this case the change of variables will not be by an orthogonal matrix.

15. Let A be an $n \times n$ matrix that commutes with *all* $n \times n$ matrices, so $AB = BA$ for all matrices B. Show that $A = cI$ for some scalar c. [SUGGESTION: Let $\vec{v}$ be an eigenvector of A with eigenvalue λ].

    SOLUTION: For this we need a preliminary result that is almost obvious.

    Let $\vec{e}_1 = (1, 0, 0, \dots, 0) \in \mathbb{R}^n$ and let $\vec{v}$ and $\vec{w}$ be any non-zero vectors in $\mathbb{R}^n$. Then there is an invertible matrix B with $B\vec{e}_1 = \vec{w}$.

    To see this, let the first column of B be the vector $\vec{w}$ and for the remaining columns use any vectors that extend $\vec{w}$ to a basis for $\mathbb{R}^n$. For instance, if the first component of $\vec{w}$ is not zero, you can use the standard basis vectors $\vec{e}_2, \dots, \vec{e}_n$ for the remaining columns of B.

More generally, *there is an invertible matrix* M *with* $M\vec{v} = \vec{w}$. This is now easy. Let A be an invertible matrix that maps $\vec{e}_1$ to $\vec{v}$. Then let $M := BA^{-1}$.

Now back to the original problem. Let $\vec{v}$ be an eigenvector of A with eigenvalue λ. Then $A\vec{v} = \lambda\vec{v}$. But then

$$A(B\vec{v}) = BA\vec{v} = \lambda(B\vec{v}).$$

In other words, every vector of the form $B\vec{v}$ for some invertible matrix B is an eigenvector of A with the same eigenvalue λ. But in the preliminaries we showed that given any non-zero vector $\vec{w}$ there is an invertible B such that $\vec{w} = B\vec{v}$.

[Last revised: December 10, 2012]